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THE ORDER OF MAGNITUDE OF THE FOURIER COEFFICIENTS IN FUNCTIONS HAVING ISOLATED SINGULARITIES

GORDON RAISBECK, Bell Telephone Laboratories

Raphael Salem has shown* how a Fourier sine or cosine series with monotonically decreasing coefficients obeying appropriate uniformity and convexity conditions can be summed approximately in the neighborhood of the origin.

The results of this paper are a qualified converse of Salem's results. Salem shows that if the Fourier coefficients of a function tend to zero sufficiently smoothly, then the function has a singularity of a type which can be associated directly with the rate at which the coefficients tend to zero. The present paper shows that if a function has a single singularity and is sufficiently regular elsewhere, then the Fourier coefficients fall off in a way which can be related directly to the nature of the singularity.

The problem is of practical interest when Fourier series are used for numerical computation. It is well known that a piecewise analytic function having a simple discontinuity has Fourier coefficients which decrease like $1/n$, where n is the order of the coefficient.† This result can be extended to functions whose derivatives have simple discontinuities. The ideas of this paper are centered about the result that a function (odd or even) which behaves like x^α at the origin and which is sufficiently regular elsewhere has Fourier coefficients which behave like $n^{-1-\alpha}$. There is no optimum way to state the result. In Theorem I the hypotheses are relaxed, and the result is less precise: $f(x)$ is assumed to be simply monotonic and convex, and its Fourier sine coefficients are shown to be bounded by a pair of definite integrals. In Theorem II, the hypotheses are restricted, and more is proved: $f(x)$ is assumed to be x^α, and its Fourier sine coefficients are shown to approach asymptotically a multiple of $n^{-1-\alpha}$. If more is assumed, more can be deduced, and contrariwise.

The oddness or evenness of the function in question is taken care of by applying the theorems to the odd part of the function and to the derivative of the even part (or the integral of the even part) which is itself odd. Hence a discussion of the integral $\int f(x) \sin nx dx$ suffices to cover all cases. The restriction of the singularity to the origin is of no consequence, because $(a_n^2+b_n^2)$ is invariant to a shift in the origin.

Here are the statement and proof of Theorem I.

---

* *Essais sur les Series Trigonometriques*, Paris 1940; also *Comptes Rendus*, Acad. Sci. Paris, v. 186, p. 1804.

† Whittaker and Watson, *Modern Analysis*, Cambridge, 1940, pp. 167–168.

THEOREM I. *Suppose $f(x)$ is defined for $0<x<\pi$. Suppose $f(x)$ is positive, non-increasing, and convex* in $(0, \pi)$, and suppose $xf(x)$ is integrable in $(0, \pi)$ and that $f(x)$ is not a constant in $(0, \pi)$. Then*

$$\int_0^\pi f(x) \sin nx dx < n \int_0^{\pi/n} xf(x)dx,$$

$$\int_0^\pi f(x) \sin nx dx > \frac{2n}{\pi} \int_0^{\pi/2n} x[f(x) + (-1)^n f(\pi - x)]dx > cn \int_0^{\pi/2n} xf(x)dx,$$

where c is a constant independent of n. If n is restricted to sufficiently large values, c may be given a value arbitrarily close to

$$c_0 = \frac{2}{\pi}\left[1 - \lim_{u \to 0} \frac{f(\pi - u)}{f(u)}\right].$$

To prove Theorem I, let

$$\int_0^\pi f(x) \sin nx dx = I = \left[\int_0^{\pi/n} + \int_{\pi/n}^{2\pi/n} + \cdots \int_{(n-1)\pi/n}^{\pi}\right] f(x) \sin nx dx$$
$$= I_0 - I_1 + I_2 - \cdots + (-1)^{n-1} I_{n-1},$$

where

$$I_m = \int_0^{\pi/n} f(x + m\pi/n) \sin nx dx.$$

Because of the monotonic character of $f(x)$,

$$I_m \leqq I_{m-1}$$

and the above sum is the sum of non-increasing sequence of terms of alternating signs. Hence

$$I \leqq I_0.$$

From the property

$$\sin nx < nx, \qquad 0 < x < \frac{\pi}{n},$$

it follows that

$$I < n \int_0^{\pi/n} xf(x)dx.$$

---

* A measurable function $f(x)$ is called convex if $f((x+y)/2) \leqq (f(x)+f(y))/2$ for any x and y in the range under consideration. An equivalent statement is that every chord lies above the curve $y=f(x)$. It follows that if $f'(x)$ exists, it is a non-decreasing function; that $f(x+a)-f(x-a)$ is a nondecreasing function of x; and also that f is continuous in an interval where it is convex, except possibly at the end points.

which is half of Theorem I.

To prove the other half of the theorem, make the alternative subdivision of the integral

$$I = \left[\int_0^{\pi/2n} + \int_{\pi/2n}^{3\pi/2n} + \int_{3\pi/2n}^{5\pi/2n} + \cdots + \int_{(2n-1)\pi/2n}^{\pi}\right] f(x) \sin nx dx$$

$$= I_0' + I_1 - I_2 + I_3 - \cdots + (-1)^{n-1} I_{n-1} + (-1)^n I_n',$$

where I is defined as before and

$$I_0' = \int_0^{\pi/2n} f(x) \sin nx dx$$

$$I_n' = \int_0^{\pi/2n} f(\pi - x) \sin nx dx$$

$$I_m = \int_0^{\pi/2n} \left[f\left(\frac{m\pi}{n} - x\right) - f\left(\frac{m\pi}{n} + x\right)\right] \sin nx dx.$$

Because of the monotonicity of $f(x)$ the I_m are positive, and because of the convexity of $f(x)$ the I_m form a non-increasing sequence. Hence

$$I \geqq I_0' - I_n'.$$

From the property

$$\sin nx > \frac{2n}{\pi} x, \qquad 0 < x < \frac{\pi}{2n},$$

it follows that

$$I > \frac{2n}{\pi} \int_0^{\pi/2n} x[f(x) + (-1)^n f(\pi - x)] dx.$$

This almost completes the proof of Theorem I. The remaining statements, involving the elimination of the second term in the integral, follow easily from the monotonicity and convexity of f. Note that in Theorem I convexity is not required to find the upper bound of the integral. Hence if the maximum order of magnitude is all that is required, monotonicity in $f(x)$ is sufficient. Some auxiliary condition is required for the lower bound to hold, however. For example, the function

$$f(x) = 1, \qquad 0 < x \leqq \pi/2$$

$$= 0, \qquad \pi/2 < x < \pi$$

satisfies all the requirements of Theorem I except convexity, but

$$\int_0^\pi f(x) \sin 4mx dx = 0$$

for all integral m.

By applying Theorem I to integrals or derivatives of $f(x)$ and integrating by parts, one can derive an endless series of results. For example:

COROLLARY. *Suppose $f(x)$ is differentiable as well as absolutely integrable in $(0, \pi)$, and suppose that $f'(x)$ satisfies the hypotheses of Theorem I. Then*

$$\int_0^\pi f(x) \cos nx dx > -\int_0^{\pi/n} xf'(x)dx,$$

$$\int_0^\pi f(x) \cos nx dx < -c\int_0^{\pi/2n} xf'(x)dx,$$

where c is a positive constant independent of n.

If the first integral is integrated by parts, the proof becomes obvious. The proof is facilitated by assuming that $\lim_{x\to 0} xf(x) = 0$, but this actually is implied by the other hypotheses.

For practical computation, there is a special result which is often useful.

THEOREM II.* *If $-1 < a < 1$,*

$$\int_0^\pi x^a \cos mx dx \sim m^{-1-a}\Gamma(a+1) \cos \frac{1}{2}\pi(a+1),$$

where $\sim$ signifies asymptotic equality, i.e., that the ratio of the two sides approaches 1 as m approaches ∞.

If $-2 < a < 0$,

$$\int_0^\pi x^a \sin mx dx \sim m^{-1-a}\Gamma(a+1) \sin \frac{1}{2}\pi(a+1).$$

In practical cases a function can often be written as the sum of x^a and a function whose Fourier coefficients are known to decrease rapidly. In such a case the Fourier coefficients of the first-named function may dominate, and hence an asymptotic expression for the coefficients is possible. An example of such a case is quoted later.

One should not conclude that under the hypotheses of Theorem I there exists an asymptotic equality of the type

$$n\int_0^{1/n} xf(x)dx \sim C\int_0^\pi f(x) \sin nx dx.$$

* Titchmarsh, *The Theory of Functions*, p. 162, problem 10.

The reader may convince himself by experimenting with functions built up of straight line segments that in fact the ratio of the two integrals may be made to oscillate within bounded limits, and hence that in the general case no asymptotic equality holds.

All these results are conveniently recalled by the mnemonic scheme mentioned earlier, *viz.*, a function (odd or even) which behaves like x^a near the origin and is sufficiently regular elsewhere has Fourier coefficients which fall off like n^{-1-a}. The reader can now see why it is not convenient to make that statement more precise: the auxiliary conditions depend on a, the oddness or evenness of the function, and so forth, while the exact nature of the result depends on how nearly the function mimics x^a. For any particular function it is often easier to proceed directly than to have recourse to such an involved general theorem, provided the nature of the anticipated result is known.

As an example of how the theorems are applied, consider the coefficients of the Fourier expansion of a cycloid

$$x = \theta - \sin\theta$$
$$f(x) = 1 - \cos\theta.$$

This is an even function of period 2π, and hence its expansion is of the form

$$f(x) = a_0/2 + \sum_1^\infty a_m \cos mx,$$

where

$$a_m = \frac{2}{\pi}\int_0^\pi f(x)\cos mx\,dx$$
$$= -\frac{2}{\pi m}\int_0^\pi f'(x)\sin mx\,dx$$

after integration by parts. It is easy to verify that

$$f'(x) = ctn\,\frac{\theta}{2}$$
$$f''(x) = -(1-\cos\theta)csc^2\frac{\theta}{2}$$

and that $-f'(x)$ is therefore monotonic and convex. Hence Theorem I applies, and the order of magnitude of a_m is between those of the two integrals

$$\int_0^{\pi/2m} xf'(x)dx, \qquad \int_0^{\pi/m} xf'(x)dx.$$

In this region $f'(x)$ can be approximated by a power of x

$$f'(x) = ctn\frac{\theta}{2}$$

$$= \frac{2}{\theta} + O(\theta)^*$$

$$= 2(3x)^{-1/3} + O(x^{1/3})$$

and

$$\int_0^{k/m} xf'(x)dx = k_1 m^{-5/3} + O(m^{-7/3}).$$

Finally

$$-cm^{-5/3} \leqq a_m \leqq -Cm^{-5/3}$$

for all sufficiently large m, where c and C are positive constants.

A shorter way to achieve the same result is through Theorem II. Observe that

$$f(x) = \frac{(6x)^{2/3}}{2} + g(x),$$

where $g(x)$ is a function with a derivative of bounded variation. The Fourier coefficients of $g(x)$ are $O(m^{-2})$. If the two functions on the right are expanded separately and the expansions added, it follows from Theorem 2 that

$$a_m \sim m^{-5/3}\left[\frac{2}{\pi}\,\Gamma\left(\frac{5}{3}\right)\cos\frac{5\pi}{6}\right]$$

where $\sim$ signifies asymptotic equality. This result is more precise than the foregoing, but the method is not applicable to as broad a class of functions.

The inspiration and constructive criticism of R. W. Hamming are acknowledged with gratitude.

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* $u = O(v)$ means that in a certain neighborhood u/v is bounded. In this case the neighborhood is that around $x = 0$.

CORRECTION

In *Accessory linkages which have certain stabilizing properties* by E. I. Gale, this MONTHLY, February, 1955, in the first line on page 97 "larger curvatures" should read "smaller curvatures".

ON THE NUMBER OF VANISHING TERMS IN AN INTEGRAL CUBIC RECURRENCE

MORGAN WARD, California Institute of Technology

Introduction. Let

$$(T): T_0, T_1, T_2, \cdots, T_n, \cdots$$

be an integral cubic recurrence; that is, the initial values T_0, T_1, T_2 of (T) are integers and

$$T_{n+3} = PT_{n+2} - QT_{n+1} + RT_n \qquad (n = 0, 1, \cdots).$$

Here P, Q and R are fixed integers and $R \neq 0$. The polynomial

$$f(z) = z^3 - Pz^2 + Qz - R$$

and the recurrence (T) are said to be associated. If

$$g(z) = T_0z^2 + (T_1 - PT_0)z + (T_2 - PT_1 + QT_0),$$

then for $|z|$ sufficiently large,

$$g(z)/f(z) = \sum_1^\infty T_{n-1}z^{-n}$$

is a generating function for (T).

The two most interesting cases are when $g(z) = df(z)/dz$ and $g(z) = 1$; we denote the corresponding recurrences by (S) and (H).

If $f(z)$ has distinct roots u, v, w, then S_n is simply the Newtonian sum of the nth powers of the roots, while H_{n+2} as a symmetric function is the homogeneous product sum of the roots of degree n [4]:

$$S_n = u^n + v^n + w^n, \qquad H_{n+2} = \sum u^{r_1}v^{r_2}w^{r_3}.$$

Here the second sum is extended over all non-negative integers r_i such that $r_1+r_2+r_3=n$. (H) is of particular importance in the algebra of recurring sequences [1], [2]; the corresponding recurrences of order two are the well known Lucas functions [3].

An integer $k \geqq 0$ is called a "zero" of the recurrence (T) if $T_k = 0$. For example, 0 and 1 are zeros of (H). We determine here the maximum number of possible zeros of (T) when its associated polynomial has integral roots subject to a restriction to be described presently.

Prior work on this problem has been confined to the cases when $f(z)$ has complex roots and $R = \pm 1$ [6], [7] or when $(T) = (S)$ and $P = 0$ [8]. There is an important general result of Kurt Mahler's [5] in this connection. Let us call both (T) and its polynomial $f(z)$ "degenerate" or "nondegenerate" according as the ratio of any pair of different roots of $f(z)$ is, or is not, a root of unity.

Mahler showed that if (T) is non-degenerate, $|T_n|$ tends to infinity with n. Hence

The total number of zeros of any non-degenerate recurrence (T) is finite.

We prove here the following more precise result.

THEOREM 1. *If the associated polynomial of an integral cubic recurrence is both non-degenerate and has integral roots which are co-prime in pairs, then at most three terms of the recurrence can vanish.*

The exact determination of the number of zeros of a given recurrence (T) under these hypotheses may be very difficult. For example, it is easy to see that (S) can have at most one zero; but the assertion that if $S_1 \neq 0$, (S) has no zeros is essentially Fermat's last theorem. On the other hand, it follows from the results of this paper that the sequence (T) with initial values $T_0=0$, $T_1=1$ and $T_2=0$ has no other zeros.

The plan of the paper is sufficiently indicated by the section headings. In the conclusion we mention some unsolved problems concerning the zeros of (H) suggested by the investigation.

2. Preliminary lemmas. We denote the roots of $f(z)$ by u, v and w. Then u, v and w are integers co-prime in pairs with distinct absolute values, since $f(z)$ is non-degenerate.

The general term T_n of (T) is of the form

$$T_n = Uu^n + Vv^n + Ww^n \tag{2.1}$$

where U, V, W are rational and different from zero, since (T) is of order three. Consequently, k and l are zeros of (T) if and only if

$$\begin{aligned} Uu^k + Vv^k + Ww^k &= 0 \\ Uu^l + Vv^l + Ww^l &= 0. \end{aligned} \tag{2.2}$$

On solving (2.2) for the ratios $U:V:W$, we obtain

LEMMA 2.1. *If k and l are distinct integers and $l>k\geq 0$, a necessary and sufficient condition that k and l be zeros of (T) is that U, V and W of formula* (2.1) *satisfy the conditions*

$$\frac{Uu^k}{w^{l-k} - v^{l-k}} = \frac{Vv^k}{u^{l-k} - w^{l-k}} = \frac{Ww^k}{v^{l-k} - u^{l-k}}.$$

We note for future use two simple corollaries of this result.

LEMMA 2.2. *If $T_0=0$, then a necessary and sufficient condition that $T_n=0$ for $n>0$ is that*

$$\frac{U}{w^n - v^n} = \frac{V}{u^n - w^n} = \frac{W}{v^n - u^n}. \tag{2.3}$$

LEMMA 2.3. *If $T_0=0$ and if $l>k>0$, then a necessary condition that $T_l=T_k=0$ is that*

$$\frac{w^k-v^k}{w^l-v^l}=\frac{u^k-w^k}{u^l-w^l}=\frac{v^k-u^k}{v^l-u^l}. \tag{2.4}$$

LEMMA 2.4. *Let t be a real variable and n any real number greater than one. Then the function $(t^n-1)/(t+1)$ increases steadily if $t>0$ and $(t^n+1)/(t+1)$ increases steadily if $t>1$.*

For the derivatives of the functions are positive under the stated conditions.

LEMMA 2.5. *Let r and s be co-prime integers and k and l positive integers. Then*

$$(r^k-s^k, r^l-s^l)=r^d-s^d \qquad \text{where } d=(k,l). \tag{2.5}$$

Here (x, y) denotes the greatest common divisor of the integers x and y.

For denote the left side of (2.5) by m. Since d divides k and l, r^d-s^d divides r^k-s^k and r^l-s^l. Consequently, r^d-s^d divides m. It suffices then to show that m divides r^d-s^d.

Since m divides r^k-s^k, it is prime to both r and s. Hence there exists a positive integer t with the property that m divides r^t-s^t but m does not divide r^n-s^n if $0<n<t$. Then *m divides r^n-s^n if and only if t divides n*. For let $n=2qt\pm c$, where $0\leqq c<t$ and let $a=qt$, $b=qt\pm c$. Then if $b\geqq a$, $r^n-s^n=(r^a-s^a)(r^b-s^b)+(rs)^a(r^c-s^c)$, and if $a\geqq b$, $r^n-s^n=(r^a-s^a)(r^b-s^b)-(rs)^b(r^c-s^c)$. In either case, since m divides r^n-s^n and r^a-s^a and is prime to r and s, the minimal property of t is contradicted unless $c=0$ or $c=t$.

Now m divides both r^k-s^k and r^l-s^l. Hence t divides both k and l. Therefore, t divides d, and m divides r^d-s^d, completing the proof.

3. Recurrences with three zeros. Let (T) be a recurrence with $T_0=0$ and at least two other zeros, k and l. We may assume that

$$0<k<l, \qquad T_n\neq 0, \qquad 0<n<k. \tag{3.1}$$

Then by Lemma 2.3, the equalities (2.4) must hold. Let $d=(k, l)$. Since u, v and w are co-prime in pairs, if we divide the numerator and denominator of each of the fractions in (2.4) by the corresponding integers w^d-v^d, u^d-w^d and v^d-u^d, we obtain by Lemma 2.5 three equal fractions in their lowest terms. Hence corresponding numerators and denominators must be equal up to sign; that is

$$\begin{aligned}\frac{w^k-v^k}{w^d-v^d}&=\pm\frac{u^k-w^k}{u^d-w^d}=\pm\frac{v^k-u^k}{v^d-u^d},\\ \frac{w^l-v^l}{w^d-v^d}&=\pm\frac{u^l-w^l}{u^d-w^d}=\pm\frac{v^l-u^l}{v^d-u^d}.\end{aligned} \tag{3.2}$$

Consider now the equalities

$$\frac{w^n - v^n}{w - v} = \pm \frac{u^n - w^n}{u - w} = \pm \frac{v^n - u^n}{v - u}, \qquad n \geqq 1. \tag{3.3}$$

Observe that each of the equalities (3.2) may be put into this form by letting $u' = u^d$, $v' = v^d$, $w' = w^d$, taking n equal to k/d or l/d and then dropping the primes.

Let a, b and c be the absolute values of u, v and w respectively. We may evidently assume that

$$w = c > b > a > 0; \qquad v = \pm b, \; u = \pm a. \tag{3.4}$$

For changing the signs of all the roots of $f(z)$ merely multiplies T_n by $(-1)^n$. There are four cases according to the choices of sign of u and v.

Case	*Roots of* $f(z)$			*Equalities* (3.3)
	u	v	w	
1	a	b	c	$\dfrac{c^n - b^n}{c - b} = \dfrac{c^n - a^n}{c - a} = \dfrac{b^n - a^n}{b - a}$
2	$-a$	b	c	$\dfrac{c^n - b^n}{c - b} = \dfrac{c^n - (-a)^n}{c + a} = \dfrac{b^n - (-a)^n}{b + a}$
3	a	$-b$	c	$\dfrac{c^n - (-b)^n}{c + b} = \dfrac{c^n - a^n}{c - a} = \dfrac{b^n - (-a)^n}{b + a}$
4	$-a$	$-b$	c	$\dfrac{c^n - (-b)^n}{c + b} = \dfrac{c^n - (-a)^n}{c + a} = \dfrac{b^n - a^n}{b - a}$

Both case 1 *and case* 2 *are impossible if* $n>1$. In case 1, this statement is evident, since $c>b>a>0$. In case 2, if $n>1$, we have

$$\frac{c^n \pm a^n}{c + a} = \frac{b^n \pm a^n}{b + a}.$$

Now let $c=ax$ and $b=ay$. Then

$$\frac{x^n \pm 1}{x + 1} = \frac{y^n \pm 1}{y + 1} \qquad \text{with } x > y > 1 \text{ and } n > 1$$

contradicting Lemma 2.4.

Both cases 3 *and* 4 *are impossible if* n *is even. For assume* n is even. Then $n>1$, and in case 3 we have

$$\frac{b^n - a^n}{b + a} = \frac{c^n - a^n}{c - a}.$$

But $(b^n-a^n)/(b+a)<(b^n-a^n)/(b-a)<(c^n-a^n)/(c-a)$ since $c>b>a>0$.

In case 4, we have

$$\frac{c^n - b^n}{c + b} = \frac{c^n - a^n}{c + a}.$$

Hence if $b=cx$ and $a=cy$, then

$$\frac{1 - x^n}{1 + x} = \frac{1 - y^n}{1 + y} \qquad \text{with } 1 > x > y > 0.$$

But by Lemma 4, $(1-t^n)/(1+t)$ steadily decreases if $t>0$.

We now apply these results to the equalities (3.2) in accordance with the remark following (3.3). First, *d must be odd.* For if d is even, u^d, v^d, w^d are positive, and case 1 applies; that is, $n=1$ so that $k=d$ and $l=d$ contrary to (3.1). Since n must be odd, k and l *must both be odd.* Hence w^k-v^k and w^d-v^d are of like sign. Therefore, the first equality (3.2) may be written as

$$\frac{w^k - v^k}{w^d - v^d} = \frac{u^k - w^k}{u^d - w^d} = \frac{v^k - u^k}{v^d - u^d}.$$

But since $T_0=T_k=0$, this equality and lemma 2.2 imply that

$$\frac{U}{w^d - v^d} = \frac{V}{u^d - w^d} = \frac{W}{v^d - u^d}.$$

Hence $T_d=0$ by Lemma 2.2. But $0<d\leqq k$. Hence $d=k$ by condition (3.1). We have thus proved:

THEOREM 2. *Let (T) be an integral cubic recurrence whose associated polynomial is non-degenerate and has integral roots which are co-prime in pairs, and assume that the first two zeros of (T) are 0 and k. Then if k is even, (T) has no other zeros. If k is odd, any other zero of (T) must be an odd multiple of k.*

It follows immediately that the recurrence with initial values 0, 1, 0 has no other zeros. On the other hand, the recurrence (H) of the introduction has $H_3=P=0$ if $u+v+w=0$. *Hence there exist recurrences with three zeros.*

4. Proof of Theorem 1. We will now use Theorem 2 to give a proof by contradiction of Theorem 1. Let (T) and $f(z)$ satisfy the hypotheses of the theorem, and suppose (T) has more than three zeros. Let the first four zeros of (T) be k_1, k_2, k_3, k_4, so that $0\leqq k_1<k_2<k_3<k_4$.

The recurrence (T') defined by $T'_n=T_{n+k_1}$ is associated with $f(z)$ and its first three zeros are 0, k_2-k_1, and k_3-k_1. Both k_2-k_1 and k_3-k_1 are odd by Theorem

2. Hence their difference k_3-k_2 is even. The recurrence (T'') defined by $T_n''=T_{n+k_2}$ is associated with $f(z)$ and its first three zeros are 0, k_3-k_2, and k_4-k_2. But k_3-k_2 is even, contradicting Theorem 2.

5. Conclusion. It follows from Theorem 2 that if (T) has three zeros, say k_1, k_2 and k_3, then $d=k_2-k_1$ is odd and the zeros lie in a recurrence (T^*) defined by $T_n^*=T_{k_1+nd}$ whose associated polynomial $f^*(z)$ has for its roots the dth powers of the roots of $f(z)$. Since the initial values of (T^*) are 0, 0, and T_2^*, $T_n^*=T_2^*H_n^*$ where (H^*) is the recurrence associated with $f^*(z)$ discussed in the introduction. We are thus led to consider the zeros of the recurrence (H).

We have seen that H_3 can vanish. The next possibility is that $H_5=0$, giving the diophantine equation

$$(u+v+w)(u^2+v^2+w^2)+uvw=0. \tag{5.1}$$

Trivial solutions of (5.1) evidently violate our hypotheses on $f(z)$. Whether or not (5.1) has non-trivial solutions appears to be unknown. The more general question of whether H_{2n+1} can vanish when $n>1$ appears to be of a difficulty comparable to the Fermat problem for the recurrence (S).

The simplest case when $(T)\neq(H)$ would have three zeros would be when $d=3$ and $k_3=k_1+5d$ making $H_5^*=0$. The related diophantine equation is obtained from (5.1) by replacing u, v, w by their cubes. It would imply then that the diophantine system

$$u^3+v^3+w^3=4^\epsilon s^3, \qquad u^6+v^6+w^6=2^\epsilon t^6, \qquad \epsilon=0 \text{ or } 1,$$

has a non-trivial solution, which appears unlikely.

It is tempting to conjecture in view of these remarks that under our hypotheses on $f(z)$, (H) is the *only* recurrence which can have more than two zeros. It follows of course from Lemma 2.2 that there exist recurrences (T) with arbitrarily assigned zeros k and l. But it may be shown by simple examples that neither Theorem 1 or Theorem 2 is true if we change our hypotheses on the roots of $f(z)$.

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ON THE MINIMALITY OF THE VARIATIONAL PRINCIPLES OF CLASSICAL PARTICLE MECHANICS

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1. Introduction. We shall be concerned here mainly with the variational principles which are associated with the names of Hamilton, Jacobi, Maupertuis and Hilbert. Each principle involves a functional in the form of an integral. The requirement of the vanishing of the first variation of the functional provides the differential equations characterizing the motion. For this reason these principles are called variational or stationary principles. If, in addition, the actual motion minimizes the functional, then the principle is called a minimum principle. We shall show here that some of the classical variational principles are minimum principles under certain conditions and only stationary principles under other conditions; others are always minimum principles, while still others are always only stationary principles. Some facts of this type are well known,* but it is hoped that the treatment presented here not only provides more complete results, but also is more elementary in method, simpler in development and yields results in more convenient form.

2. Notation having a fixed significance throughout the remainder of this paper. We shall consider a system $\mathfrak{M}$ of N particles. Let the configuration of $\mathfrak{M}$ be described by $n=3N$ Cartesian coordinates $[z_1, z_2, \cdots, z_n]$ in the usual manner; *i.e.*, $z_{3r-2}, z_{3r-1}, z_{3r}$ are the coordinates of the r-th particle ($r=1, 2, \cdots, N$). The configuration of $\mathfrak{M}$ is then represented by a point $z=\{z_1, z_2, \cdots, z_n\}$ in n-dimensional Euclidean space E_n. The following summation convention will be used: an index appearing more than once in the same term will be understood to be summed over its range unless otherwise specified. The range of i and j will be $[1, 2, \cdots, n]$; the range of σ will be $[1, 2, \cdots, k]$, where k will be introduced below. Unless otherwise stated it will be understood that any statement in which any of these indices appear unsummed, applies when such indices are assigned each value in their range.

The mass of the r-th particle ($r=1, 2, \cdots N$) is denoted by $m_{3r-2}=m_{3r-1}=m_{3r}$, and $m=\min_i m_i$ while $M=\max_i m_i$.

We shall suppose that $\mathfrak{M}$ is subjected to certain forces, the details of which will be considered subsequently; under the influence of these forces the configuration of $\mathfrak{M}$ during the time interval D: $t_0 \leqq t \leqq t_1$ is described by the E_n-valued function $x(t)=\{x_1(t), x_2(t), \cdots, x_n(t)\}$. Let $x(t_0)=x^0$ and $x(t_1)=x^1$. We shall suppose that the origin of the coordinate system has been chosen at x^0 and that $t_1-t_0=\delta>0$. The function x will be called the *actual motion* or the *solution*. If Δ is any time interval ($\tau_0 \leqq t \leqq \tau_1$) then any E_n-valued function $\xi(t)=\{\xi_1(t), \xi_2(t), \cdots, \xi_n(t)\}$ having a continuous derivative on Δ and such that $\xi(\tau_0)=x^0$ and $\xi(\tau_1)=x^1$ will be called a *Δ-motion of* $\mathfrak{M}$. A region Σ of $E_n \times D$

* *Cf. e.g.*, E. T. Whittaker, *Analytical Dynamics*, Dover 1944, pp. 250–253; also W. F. Osgood, *Mechanics*, Macmillan 1937, pp. 381–388.

will be called a *tube about the solution* if (a) for each t in D, $(x(t), t)$ is in Σ, and (b) for each (z, t) in Σ and each ϵ in $0 \leqq \epsilon \leqq 1$, $(x(t)+\epsilon[z-x(t)], t)$ is in Σ. If ξ is a D-motion and if, for each t in D, $(\xi(t), t) \in \Sigma$ then we say that ξ *lies in* Σ.

The following notational abbreviations will also be used. If g is a function of the ν variables, $y_1, y_2, \cdots, y_\nu$, and $y = \{y_1, y_2, \cdots, y_\nu\}$ then we shall write $\partial g(y)/\partial y_\mu = g_{,\mu}(y)$, $\partial^2 g(y)/\partial y_\mu \partial y_\rho = g_{,\mu\rho}(y)$ *etc.*; if $u(t) = \{u_1(t), u_2(t), \cdots, u_n(t)\}$ is an E_n-valued function on D then $[\int_{t_0}^{t_1} u_i u_i dt]^{1/2}$ is written as $\|u\|$.

The forces that we shall consider stem from three sources: (i) from a potential function $\Phi(z, v, t)$ dependent on the position, velocity and time [the force associated with this is $-(\partial\Phi/\partial z_i)+(d/dt)/(\partial\Phi/\partial v_i)$]; (ii) from constraints $\phi^\sigma(z, v, t) = 0$. If it should happen that no constraints are present we take $k = 0$ so that σ ranges over the null set and the terms involving ϕ^σ may be considered absent. We assume that Φ and ϕ^σ are continuous and have two continuous and bounded derivatives with respect to their first $2n$ real arguments for all (z, t) in some tube S, about the solution, and for all values of v; (iii) forces which are due neither to the generalized potential nor to the constraints. These will be denoted by $\{F_i(t; \xi)\}$. For each t, $F_i(t; \xi)$ is a functional defined over a subset of D-motions which includes x. We assume that there exist k functions $\{\lambda_\sigma(t)\}$ which together with the n components of the solution $\{x_i(t)\}$ satisfy the n Lagrange equations

$$(1) \qquad \frac{d}{dt}[m_i\dot{x}_i - \Phi_{,n+i}(Q) + \lambda_\sigma\phi^\sigma_{,n+i}(Q)] + \Phi_{,i}(Q) - \lambda_\sigma\phi^\sigma_{,i}(Q) = F_i(t; x),$$

(no sum on i) and the k constraint equations $\phi^\sigma(Q) = 0$, where $Q = (x(t), \dot{x}(t), t)$.

The following notation will be used to specify the force situation under consideration. $\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3, \mathcal{U}_4$ signifies that $\Phi(z, v, t)$ is respectively $V(z, v, t)$, $V(v, t)$, $V(z, t)$, $V(z)$; $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ signifies that $\phi^\sigma(z, v, t)$ is respectively $f^\sigma(z, v, t)$, $f^\sigma(z, t)$, workless; $\mathcal{F}_1, \mathcal{F}_2$ signifies that $F_i(t; \xi)$ are present or absent, respectively. Thus, for example, the notation $\mathcal{U}_4\mathcal{C}_3\mathcal{F}_2$ designates the conservative case.

3. Lemmas. We give now some simple inequalities which will be used subsequently.

LEMMA 1. *Let* $u(t) = \{u_1(t), u_2(t), \cdots, u_n(t)\}$ *be continuous and have a sectionally continuous derivative on* D. (a) *If* $u(t_0) = u(t_1) = 0$, *then* $\|\dot{u}\| \geqq \pi\|u\|/\delta$; (b) *if only* $u(t_0) = 0$, *then* $\|\dot{u}\| \geqq \pi\|u\|/2\delta$.

Proof: (a) Let

$$a_{i\nu} = \frac{2}{\sqrt{\delta}}\int_{t_0}^{t_1} \dot{u}_i \cos\frac{\nu\pi}{\delta}(t - t_0)dt \qquad (\nu = 1, 2, \cdots),$$

then

$$2\|\dot{u}\|^2 = 2\int_{t_0}^{t_1} \dot{u}_i\dot{u}_i dt = a_{i\nu}a_{i\nu} \geqq \frac{a_{i\nu}a_{i\nu}}{\nu^2} = \frac{2\pi^2}{\delta^2}\int_{t_0}^{t_1} u_i u_i dt = \frac{2\pi^2}{\delta^2}\|u\|^2.$$

Remark: The inequality cannot be improved, since if $u_i(t)=\sin[\pi(t-t_0)/\delta]$, then $\delta\|\dot{u}\|=\pi\|u\|$. (b) By extending $u_i(t)$ by reflection in the line $t=t_1$, the conditions of Lemma 1 are satisfied on the interval $t_0\leqq t\leqq t_1+\delta$ of length 2δ. Then

$$2\|\dot{u}\|^2=\int_{t_0}^{t_1+\delta}\dot{u}_i\dot{u}_i dt\geqq\frac{\pi^2}{(2\delta)^2}\int_{t_0}^{t_1+\delta}u_iu_i dt=\frac{2\pi^2}{4\delta^2}\int_{t_0}^{t_1}u_iu_i dt=\frac{2\pi^2}{4\delta^2}\|u\|^2.$$

Again the inequality cannot be improved, for if $u_i(t)=\sin[\pi(t-t_0)/2\delta]$ then $2\delta\|\dot{u}\|=\pi\|u\|$.

LEMMA 2. *Let $\omega(\epsilon)$ have two continuous derivatives on $0\leqq\epsilon\leqq 1$ and $\omega'(0)=0$. If $\min_{0\leqq\epsilon\leqq 1}\omega''(\epsilon)\geqq 2b$ then $\omega(0)\leqq\omega(1)-b$. If $\max_{0\leqq\epsilon\leqq 1}\omega''(\epsilon)\leqq -2c$ then $\omega(0)\geqq\omega(1)+c$.*

Proof: $\omega(1)-\omega(0)=\int_0^1\int_1^\epsilon\omega''(s)dsd\epsilon$.

4. Hamilton's principle. We assume forces of type $\mathcal{U}_1\mathcal{C}_1\mathcal{J}_1$. For each D-motion ζ define the functional

$$I(\zeta)=\int_{t_0}^{t_1}\left[\frac{1}{2}m_i\dot{\zeta}_i^2-V(\zeta,\dot{\zeta},t)+\lambda_\epsilon f^\epsilon(\zeta,\dot{\zeta},t)+F_i(t;x)\zeta_i\right]dt.$$

Now let X be an arbitrary, but fixed D-motion. If we can show that $I(x)\leqq I(X)$ we have a minimum principle. [If there are no forces F_i present and if we require that X satisfy the constraints, then $I(X)$ and $I(x)$ become $\int_{t_0}^{t_1}(T-V)dt$ evaluated along the respective motions (T is the kinetic energy); *i.e.*, if one shows that, for arbitrary $\zeta\neq x$, $I(x)<I(\zeta)$ then $\int_{t_0}^{t_1}(T-V)dt$ is less when computed along the actual motion than when computed along any other path satisfying the constraints.] Let $\alpha=X-x$, and for $0\leqq\epsilon\leqq 1$ define $\omega(\epsilon)=I(x+\epsilon\alpha)$. Now $\omega(0)=I(x)$, $\omega(1)=I(X)$, while from equation (1) and the fact that $\alpha(t_0)=\alpha(t_1)=0$ it follows that $\omega'(0)=0$. Thus Lemma 2 will allow us to establish the results we seek if we can find bounds for $\omega''(\epsilon)$. Now

$$\omega''(\epsilon)=\int_{t_0}^{t_1}\{m_i\dot{\alpha}_i^2+[\lambda_\epsilon f^\epsilon_{,ij}(R)-V_{,ij}(R)]\alpha_i\alpha_j+2[\lambda_\epsilon f^\epsilon_{,(n+i)j}(R)-V_{,(n+i)j}(R)]\dot{\alpha}_i\alpha_j+[\lambda_\epsilon f^\epsilon_{,(n+i)(n+j)}(R)-V_{,(n+i)(n+j)}(R)]\dot{\alpha}_i\dot{\alpha}_j\}dt, \tag{2}$$

where $R=(x(t)+\epsilon\alpha(t),\ \dot{x}(t)+\epsilon\dot{\alpha}(t),\ t)$, and bounds on this can be established under various conditions. We shall select some of the simple ones.

THEOREM 1a. *Let the forces be of the type $\mathcal{U}_3\mathcal{C}_2\mathcal{J}_1$ and let $c_{ij}(z,t)$ be the symmetric matrix $[V_{,ij}(z,t)-\lambda_\epsilon f^\epsilon_{,ij}(z,t)]$. Let $\mu(z,t)$ be the largest characteristic value of $[c_{ij}(z,t)]$ and let $\mu=\sup_{(z,t)\in S}\mu(z,t)$. If $\mu\leqq 0$ or, if $\mu>0$ and $\delta<\pi(m/\mu)^{1/2}$ then, for any D-motion ζ lying in S, $I(x)\leqq I(\zeta)$. Indeed*

$$I(\zeta)\geqq I(x)+\tfrac{1}{2}([m\pi^2/\delta^2]-\mu)\cdot\|\zeta-x\|^2.$$

Proof: $[V_{,ij}(z,t)-\lambda_\epsilon f^\epsilon_{,ij}(z,t)]\alpha_i\alpha_j\leqq\mu(z,t)\alpha_i\alpha_i$. By Lemma 1 $\omega''(\epsilon)\geqq\tfrac{1}{2}([m\pi^2/\delta^2]-\mu)\|\alpha\|^2$ so that Lemma 2 is applicable.

Remark: Let $C=\sup_{\chi\in s}\sqrt{c_{ij}(\chi)c_{ij}(\chi)}$. Then $\mu\leqq C$ and we may also conclude that $I(\zeta)\geqq I(x)+\frac{1}{2}(m\pi^2\delta^{-2}-C)\cdot\|\zeta-x\|^2$.

This result is sometimes phrased "Hamilton's Integral is a minimum if the duration is sufficiently short." On the other hand no fixed time interval is sufficiently short for all fields, even in the conservative case. This can be seen in the following way. Assume the forces are $\mathcal{U}_3\mathcal{C}_2\mathcal{J}_2$ with $V=\kappa\Psi(z,t)$. We shall say that the field is strengthened when κ is increased. Assume that $\Psi_{,rr}(z,t)\geqq\rho>0$ and continuous in some slab of $E_n\times D: -\infty<z_i<\infty$, $t_0\leqq\tau_0<t<\tau_1\leqq t_1$ for some r in $[1,2,\cdots,n]$. We then let $\eta_r(t)$ be a function which is continuously differentiable and zero outside of $\tau_0<t<\tau_1$ and positive inside. Let $\eta_i(t)\equiv 0$ when $i\neq r$. Let $a=\int_{t_0}^{t_1}m_r\dot{\eta}_r^2dt$ and $b=\int_{t_0}^{t_1}\eta_r^2dt$. Then $b>0$ and $\omega''(\epsilon)\leqq a-\kappa\rho b$. Thus if $\kappa>a/\rho b$ then $\omega''(\epsilon)<0$. Hence we have proved

THEOREM 1b. *Consider a field with potential $\Psi(z,t)$ such that $\Psi_{,rr}(z,t)\geqq\rho>0$ and continuous in some slab of $E_n\times D$ for some r and such that for each κ_0 there is a $\kappa>\kappa_0$ such that there is a solution $\xi(t)$ to the equations $m_i\ddot{\xi}_i+\kappa\Psi_{,i}(\xi,t)=0$ (no sum on i), $\xi(t_0)=x^0$, $\xi(t_1)=x^1$. Then it is always possible to strengthen the field and find a D-motion $x+\eta$ such that $I(x)>I(x+\epsilon\eta)$ for every $\epsilon>0$.*

It is impossible to state this as a maximum principle since an examination of equation (2) for the present case and consideration of the fact that it is possible to find arbitrarily small functions with arbitrarily large derivatives shows that there are always many η's such that $I(x+\epsilon\eta)$ has a proper minimum when $\epsilon=0$.

As an example we may consider the motion of a mass on a spring of spring constant κ. Let x be the displacement from equilibrium, $x(t_0)=0$, $x(t_1)=x^1\neq 0$. Here $V=\frac{1}{2}\kappa x^2$. Then, by Theorem 1a, the actual motion minimizes Hamilton's Integral if $\delta^2<m\pi^2/\kappa$. On the other hand if, following Theorem 1b, we choose, *e.g.*, $\eta(t)=\sin\pi(t-t_0)/\delta$, then $\omega''(\epsilon)=m\pi^2/2\delta-\kappa\delta/2$, so that the actual motion does not minimize Hamilton's Integral if $\kappa>m\pi^2/\delta^2$. (We assume in each case that $\delta\sqrt{\kappa/m}$ is not an integral multiple of π so that the desired motion is possible.) Since the period is $2\pi\sqrt{m/\kappa}$, our result shows that the actual motion minimizes Hamilton's Integral if the time interval is less than half the period while it does not minimize the integral if the time interval exceeds half the period.

THEOREM 1c. *Let the forces be of type $\mathcal{U}_2\mathcal{C}_2\mathcal{J}_1$. For each t in D and each v in E_n let $\gamma(v,t)$, $\Gamma(v,t)$ denote the smallest and largest characteristic values of the symmetric matrix $[V_{,ij}(v,t)]$. Let $\gamma=\inf_{v,t}\gamma(v,t)$ and $\Gamma=\sup_{v,t}\Gamma(v,t)$. Let ζ be any D-motion. If $\Gamma<m$ then we have the minimum principle*

$$I(\zeta)\geqq I(x)+\frac{\pi^2(m-\Gamma)}{2\delta^2}\|\zeta-x\|^2;$$

but if $\gamma>M$ then we have the maximum principle

$$I(x) \geqq I(\xi) + \frac{\pi^2(\gamma - M)}{2\delta^2} \|\xi - x\|^2.$$

It is not difficult to construct fields satisfying either of these conditions.

5. Generalized coordinates. We now suppose that the system is described by generalized coordinates $q = (q_1, q_2, \cdots, q_n)$ with constraints $f^r(q, \dot{q}, t) = 0$. Here the transformation $H\xi = q$ is a homeomorphism H from E_n to an n-dimensional Riemannian space $\mathfrak{E}$. Each tube Σ about the solution in $E_n \times D$ is mapped into a region $H\Sigma$ in $\mathfrak{E} \times D$ and the curves $\xi(t)$ are mapped into curves $H\xi(t)$. Since the values of the kinetic energy, potential energy and forces are the same whether computed in terms of the q's or the x's (t is the same in either case), the integrals we have been considering will have the same value whether computed along $\xi(t)$ or $H\xi(t) = q(t)$. Thus the remarks made above concerning the minimality of the integral when taken along the solution are also applicable in case the computations are made in generalized coordinates.

This remark applies also to the results in the remainder of the paper (to apply it to section (7) below replace the n here by $2n$).

6. Least action principle. Here we consider only forces of type $\mathcal{U}_0\mathcal{C}_2\mathcal{J}_2$.* With this assumption equation (1) shows that the energy $\frac{1}{2}m_i\dot{x}_i^2 + V(x)$ is a constant, say h, for all t in D. For each interval Δ: $\tau_0 \leqq t \leqq \tau_1$ and each Δ-motion ξ let $J(\xi) = \int_{\tau_0}^{\tau_1} [\frac{1}{2}m_i\xi_i'^2 - V(\xi) + h]dt$. Now let $\mathfrak{D}$ be an arbitrary but fixed interval $t_0 \leqq t \leqq t_1$ with $t_1 - t_0 = \mathfrak{d} > 0$ and $\mathfrak{d} \neq \delta$ and let $\mathfrak{X}$ be an arbitrary but fixed $\mathfrak{D}$-motion. We then inquire how $J(\mathfrak{X})$ compares with $J(x)$. (Of course if we demand of $\mathfrak{X}$ that for each t in $\mathfrak{D}$

$$\frac{1}{2}m_i\dot{\mathfrak{X}}_i^2 + V(\mathfrak{X}) = h, \tag{3}$$

then we are comparing $\int 2T dt$ along the motions described by $\mathfrak{X}$ and x.)

In the integral for $J(\mathfrak{X})$ change the variable of integration from t to u by the formula $t = t_0 + \mathfrak{d}(u - t_0)/\delta$ and let $X(u) = \mathfrak{X}(t)$. Then $X(u)$ is a D-motion. Let $\alpha(t)$ be $X(t) - x(t)$ for each t in D and let

$$\omega(\epsilon) = \int_{t_0}^{t_1} \left\{ \frac{m_i(\dot{x}_i + \epsilon\dot{\alpha}_i)^2}{2(1 + \epsilon\theta)} + [h - V(x + \epsilon\alpha)][1 + \epsilon\theta] \right\} dt,$$

where $\theta = \mathfrak{d}/\delta - 1 > -1$ and $0 \leqq \epsilon \leqq 1$. Now $\omega(0) = J(x)$, $\omega(1) = J(\mathfrak{X})$ and $\omega'(0) = 0$. So again the desired results can be established by the use of Lemma 2. Now

$$\omega''(\epsilon) = \int_{t_0}^{t_1} \left\{ \frac{m_i(\theta\dot{x}_i - \dot{\alpha}_i)^2}{(1 + \epsilon\theta)^3} - 2\theta V_{,i}(x + \epsilon\alpha)\alpha_i - V_{,ij}(x + \epsilon\alpha)\alpha_i\alpha_j(1 + \epsilon\theta) \right\} dt. \tag{4}$$

Again there are many force situations for which $\omega''(\epsilon)$ can be shown to be

* The principle can be extended to the non-conservative case. *Cf.* Whittaker, *op. cit.*, p. 248.

bounded. We consider here only one simple one. Assume that $x(t)$ is not identically zero. Let $[\sup_{(s,t)\in S} \Sigma_i V_i^2(z)]^{1/2}=\Lambda$ and $\sup_{(z;t)\in S} \Omega(z)=\Omega$, where $\Omega(z)$ is the maximum characteristic value of the matrix $[V_{,ij}(z)]$. Then with the help of Lemma 1 we have

$$\omega''(\epsilon) \geqq \frac{\pi^2 m}{4}(1+\epsilon\theta)^{-3}\delta^{-2}\|\theta x-\alpha\|^2 - 2|\theta|\Lambda\sqrt{\delta}\|\alpha\| - (1+\epsilon\theta)\Omega\|\alpha\|^2;$$

hence

$$\omega''(\epsilon) \geqq \left[\frac{\pi^2 m}{4}(1+\epsilon\theta)^{-3}\delta^{-2} - (1+\epsilon\theta)\Omega\right]\|\alpha\|^2$$

$$- 2|\theta|\left[\Lambda\delta^{1/2} + \frac{\pi^2 m}{4}\|x\|(1+\epsilon\theta)^{-3}\delta^{-2}\right]\|\alpha\| + \frac{\pi^2 m}{4}\theta^2(1+\epsilon\theta)^{-3}\delta^{-2}\|x\|^2,$$

which is of the form $A\|\alpha\|^2-2B\|\alpha\|+C$ with $C>0$. Thus if $\|\alpha\|$ is sufficiently small, say less than $\mathfrak{a}$, then $\omega''(\epsilon)>0$. To put this in more physical form observe that $\|\alpha\| \leqq (n\delta)^{1/2}\sup_{i,t}|\alpha_i(t)|$. Hence if $\sup_{i,t}|\alpha_i(t)|<(\mathfrak{a})/(n\delta)^{1/2}$ we have $\omega''(\epsilon)>0$. To bring these results back to the interval $\mathfrak{D}$ let $\xi(t)=x(t_0+(\delta/\mathfrak{d})(t-t_0))$. This proves

THEOREM 2. *Let the forces be of the type $\mathcal{U}_4\mathcal{C}_2\mathcal{T}_2$ and the solution x not identically zero. Let $\mathfrak{S}$ denote the projection of S on E_n and $\mathfrak{D}$ be any interval $t_0\leqq t\leqq t_1$ with $t_1-t_0=\mathfrak{d}\neq\delta$. Let $\xi(t)=x(t_0+\delta(t-t_0)/\mathfrak{d})$, $t\in\mathfrak{D}$. Then there is a $\mathfrak{d}>0$ such that if $\mathfrak{X}(t)$ is any $\mathfrak{D}$-motion whose values lie in $\mathfrak{S}$ and satisfy the condition $\sup_{i;t\in\mathfrak{D}}|\mathfrak{X}_i(t)-\xi_i(t)|<\mathfrak{d}$ then $J(x)<J(\mathfrak{X})$.*

One might remark that, in the above discussion, if $A>0$ and $C>B^2/A$ then $\omega''(\epsilon)>0$ for all values of $\|\alpha\|$, so that the last inequality in the conditions of the above theorem as well as the conditions $\mathfrak{d}\neq\delta$ and $x\not\equiv 0$ could then be dropped.

If we require of $\mathfrak{X}(t)$ that it be a $\mathfrak{D}$-motion satisfying (3) then: (a) $J(\mathfrak{X})=\int_{t_0}^{t_1}2T\,dt$; (b) $J(\mathfrak{X})=2\int_{x^0}^{x^1}\sqrt{h-V(\mathfrak{X})}\,ds$, [where $(ds)^2=\frac{1}{2}m_i(d\mathfrak{X}_i)^2$]; (c) if also $V=0$ then $J(\mathfrak{X})=2h(t_1-t_0)$; and (d) if also $V=0$ then $J(\mathfrak{X})=2\sqrt{h}\int_{x^0}^{x^1}ds$. Thus statements about the minimality of $J(x)$ when compared to $J(\mathfrak{X})$ for any $\mathfrak{X}$ satisfying (3) are also statements about the minimality of (a) Maupertuis' Least Action Principle, (b) Jacobi's Least Energy Principle, (c) The principle of least time and (d) the geodesic principle. Indeed in the last two cases equation (4) shows immediately that $\omega''(\epsilon)\geqq 0$ so that these are truly *minimum* principles.

7. Hamilton's Modified Principle. We introduce now points $(\xi,\psi)=\{\xi_1,\xi_2,\cdots,\xi_n,\psi_1,\psi_1,\cdots,\psi_n\}$ in $E_n\times E_n$. The motion of $\mathfrak{M}$ will now be described by an $E_n\times E_n$-valued function of t: $(x(t),p(t))=\{x_1(t),x_2(t),\cdots,x_n(t),p_1(t),p_2(t),\cdots,p_n(t)\}$, $t\in D$. This we shall call the *actual phase path*. For any interval Δ: $\tau_0\leqq t\leqq\tau_1$ we shall define a *Δ-phase motion* to be any $E_n\times E_n$-valued function $(\xi(t),\psi(t))$ which has a continuous derivative on Δ and satisfies the conditions $\xi(t_0)=x^0$, $\xi(t_1)=x^1$. Let the forces be of the type $\mathcal{U}_3\mathcal{C}_3\mathcal{T}_2$. The actual

phase path is then, according to Lagrange's equations, a D-phase path satisfying the $2n$ equations

$$(5) \qquad \begin{aligned} p_i &= m_i\dot{x}_i \text{ (no sum)}, \\ \dot{p}_i &= -V_{,i}(x, t). \end{aligned}$$

For each D-phase path (ζ, π) define

$$K(\zeta, \pi) = \int_{t_0}^{t_1}\left[\pi_i\dot{\zeta}_i - \frac{\pi_i^2}{2m_i} - V(\zeta, t)\right]dt.$$

Let (X, P) be a D-phase motion. Let $\alpha = X - x$, $\beta = P - p$ and

$$\begin{aligned} \omega(\epsilon) &= K(x + \epsilon\alpha, p + \epsilon\beta) \\ &= \int_{t_0}^{t_1}\left[(p_i + \epsilon\beta_i)(\dot{x}_i + \epsilon\dot{\alpha}_i) - \frac{(p_i + \epsilon\beta_i)^2}{2m_i} - V(x + \epsilon\alpha, t)\right]dt. \end{aligned}$$

Then $\alpha(t_0) = \alpha(t_1) = 0$, $\omega(0) = K(x, p)$, $\omega(1) = K(X, P)$ and $\omega'(0) = 0$. Now

$$\omega''(\epsilon) = -\int_{t_0}^{t_1}[2\alpha_i\dot{\beta}_i + \beta_i^2/m_i + V_{,ij}(x + \epsilon\alpha, t)\alpha_i\alpha_j]dt.$$

This shows that we can always find a path (X, P) arbitrarily close to (x, p) such that $\omega''(\epsilon) > 0$ and another such that $\omega''(\epsilon) < 0$. This follows from the fact that $|\beta_i|$ can be arbitrarily small while $|\dot{\beta}_i|$ is arbitrarily large. At the same time α_i could be arbitrarily small in absolute value but with suitable variations in sign so that the first term in the integrand will determine the sign of the whole integrand and be either positive or negative at our choice. Hence

THEOREM 3. *Hamilton's Modified Principle cannot be stated as a minimum (or a maximum) principle for a conservative field regardless of the assumptions regarding the field or the length of the time interval.*

We shall consider finally Hilbert's Variational Principle. Let the forces be of type $\mathcal{U}_4\mathcal{C}_3\mathcal{J}_3$. The actual phase path is a D-phase motion satisfying the equations (5), and now also

$$\frac{p_i^2}{2m_i} + V = h.$$

For each interval Δ: $\tau_0 \leqq t \leqq \tau_1$ and any Δ-phase path (ξ, ψ) define

$$G(\xi, \psi) = \int_{\tau_0}^{\tau_1}\left[\xi_i\psi_i - \frac{\psi_i^2}{2m_i} - V(\xi) + h\right]dt.$$

Now let $\mathfrak{D}$ be an arbitrary but fixed interval $t_0 \leqq t \leqq t_1$ where $t_1 - t_0 = b > 0$ and

let $(\mathfrak{X}, \mathfrak{P})$ be an arbitrary but fixed $\mathfrak{D}$ phase-motion. In the integral defining $G(\mathfrak{X}, \mathfrak{P})$ let $t=t_0+b(u-t_0)/\delta$ and let $X(u)=\mathfrak{X}(t)$, $P(u)=\mathfrak{P}(t)$. Let $\alpha(t)=X(t)-x(t)$, $\beta(t)=P(t)-p(t)$ and

$$\omega(\epsilon)=\int_{t_0}^{t_1}(p_i+\epsilon\beta_i)(\dot{x}_i+\epsilon\dot{\alpha}_i)-[1+\epsilon\theta]\left[\frac{(p_i+\epsilon\beta_i)^2}{2m_i}+V(x+\epsilon\alpha)-h\right]dt,$$

where $\theta=b/\delta-1$.

Then $\omega(0)=G(x, p)$, $\omega(1)=G(\mathfrak{X}, \mathfrak{P})$, $\omega'(0)=0$. Now

$$\omega''(\epsilon)=-\int_{t_0}^{t_1}\left\{2\alpha_i\dot{\beta}_i+[1+\epsilon\theta]\left[\frac{\beta_i^2}{m_i}+V_{,ij}(x+\epsilon\alpha)\alpha_i\alpha_j\right]\right.$$
$$\left.+2\theta\left[\frac{(p_i+\epsilon\beta_i)\beta_i}{m_i}+V_{,i}(x+\epsilon\alpha)\alpha_i\right]\right\}dt.$$

The same sort of considerations apply here as to the previous case and so the principle can not be stated as a minimum (or a maximum) principle.

The writer wishes to thank Professors F. Koehler, A. Milgram, and C. Langenhop for helpful discussions concerning the material presented here.

ON THE REDUCTION OF A MATRIX TO DIAGONAL FORM

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Given an $n\times n$ matrix $A=(a_{ij})$ with elements in a field F, one wishes to know when it is possible to find a matrix P so that the matrix $D=PAP^{-1}$ is in diagonal form. Of course even when this is possible it will be necessary, in general, to take the elements of P and therefore of D in some extension field of F, specifically in the algebraic closure of F. Nevertheless it is desirable to have a rational criterion for the diagonalizability of A, *i.e.*, a criterion which can be verified entirely within the given field F irrespective of where the elements of P and D lie. In this paper we give such a criterion.

We shall restrict attention to the case of fields of characteristic zero, and we denote by I the identity $n\times n$ matrix.

By the degree of a matrix A we shall mean the degree of its minimal polynomial. Thus if A has degree s then $I, A, \cdots, A^{s-1}$ are linearly independent while $I, A, \cdots, A^{s-1}, A^s$ are linearly dependent over F.

THEOREM. *Let A be an $n\times n$ matrix of degree s with elements in a field F of characteristic zero. Let S denote the trace and Δ the determinant*

$$\Delta=\det[S(A^{i+j})], \qquad (0\leqq i, j\leqq s-1).$$

Then A can be reduced to diagonal form (in some extension field of F) if and only if $\Delta \neq 0$.

Proof. If A can be reduced to diagonal form then

$$PAP^{-1} = B = \begin{pmatrix} \lambda_1 I_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_r I_r \end{pmatrix}$$

where $\lambda_1, \cdots, \lambda_r$ are the distinct characteristic roots of A and I_k is the n_k-rowed identity matrix ($n_1 + \cdots + n_r = n$). Evidently $r = s$ and for $u = 1, 2, 3, \cdots$, $S(A^u) = S(B^u)$, hence

$$S(A^{i+j}) = n_1\lambda_1^{i+j} + \cdots + n_r\lambda_r^{i+j}, \qquad (0 \leqq i, j \leqq r-1).$$

This implies, by the product rule for determinants

$$\Delta = \det\,[S(A^{i+j})] = \begin{vmatrix} n_1 & n_2 & \cdots & n_r \\ n_1\lambda_1 & n_2\lambda_2 & \cdots & n_r\lambda_r \\ \cdots & \cdots & \cdots & \cdots \\ n_1\lambda_1^{r-1} & n_2\lambda_2^{r-1} & \cdots & n_r\lambda_r^{r-1} \end{vmatrix} \begin{vmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{r-1} \\ 1 & \lambda_2 & \cdots & \lambda_2^{r-1} \\ \cdots & \cdots & \cdots & \cdots \\ 1 & \lambda_r & \cdots & \lambda_r^{r-1} \end{vmatrix}$$

$$= n_1 \cdots n_r V^2,$$

where V is the (non-vanishing) Vandermonde determinant of $\lambda_1, \cdots, \lambda_r$.

Conversely, suppose A cannot be put in diagonal form. We see from the Jordan canonical form that if $\lambda_1, \cdots, \lambda_r$ are the distinct characteristic roots of A and since s is the degree of A, $s \geqq r$. As before let λ_k have multiplicity n_k so for some k, $n_k > 1$ (or else A is diagonalizable). It is then easily seen (again from the Jordan canonical form) that $(A - \lambda_1 I)(A - \lambda_2 I) \cdots (A - \lambda_r I) \neq 0$, so that $s > r$. We again have

$$S(A^{i+j}) = n_1\lambda_1^{i+j} + \cdots + n_r\lambda_r^{i+j}, \qquad (0 \leqq i, j \leqq s-1),$$

but since Δ is the determinant of an $s \times s$ matrix we have

$$\Delta = \det\,[S(A^{i+j})] = \begin{vmatrix} n_1 & \cdots & n_r & 0 \cdots 0 \\ n_1\lambda_1 & \cdots & n_r\lambda_r & 0 \cdots 0 \\ \cdots & \cdots & \cdots & \cdots \\ n_1\lambda_1^{s-1} & \cdots & n_r\lambda_r^{s-1} & 0 \cdots 0 \end{vmatrix} \begin{vmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{s-1} \\ \cdots & \cdots & \cdots & \cdots \\ 1 & \lambda_r & \cdots & \lambda_r^{s-1} \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{vmatrix} = 0.$$

Example. Let

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}.$$

Clearly the degree of A is 2. We have

$$A^2 = \begin{pmatrix} 7 & 10 \\ 15 & 22 \end{pmatrix} \quad \text{so} \quad \Delta = \begin{vmatrix} S(I) & S(A) \\ S(A) & S(A^2) \end{vmatrix} = \begin{vmatrix} 2 & 5 \\ 5 & 29 \end{vmatrix} = 33 \neq 0;$$

hence A is reducible to diagonal form. We can choose

$$P = \begin{pmatrix} -3+\sqrt{33} & 4 \\ -3-\sqrt{33} & 4 \end{pmatrix} \quad \text{and then} \quad D = \begin{pmatrix} \frac{1}{2}(5+\sqrt{33}) & 0 \\ 0 & \frac{1}{2}(5-\sqrt{33}) \end{pmatrix}.$$

The above theorem follows also from certain well-known results in the theory of linear algebras. To consider the problem from this viewpoint, we fix the matrix A and the field F as well as an algebraic closure $\overline{F}$ of F.

We say that A is reducible over F if, by a multiplication of the type QAQ^{-1}, where Q is an $n \times n$ matrix with elements in F, A can be transformed to a matrix of the form

$$\begin{pmatrix} M_1 & M_2 \\ 0 & M_3 \end{pmatrix}$$

where, for some m, $1 \leqq m \leqq n-1$, M_1 is an $m \times m$ matrix, M_3 is an $(n-m) \times (n-m)$ matrix, and M_2 is an $m \times (n-m)$ matrix. Otherwise we say that A is irreducible. We say that A is completely reducible if, by a multiplication of the above type, A can be transformed to a matrix of the type

$$\begin{pmatrix} N_1 & 0 & \cdots & 0 \\ 0 & N_2 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & N_t \end{pmatrix}$$

where each N_i is an irreducible $n_i \times n_i$ matrix ($n_1 + \cdots + n_t = n$). In particular, (for $t=1$) an irreducible matrix is completely reducible.

It can be shown that if A is irreducible over F then its minimal polynomial is irreducible over F; more generally, A is completely reducible over F if and only if its minimal polynomial is the product of distinct factors each irreducible over F, or equivalently, if and only if its minimal polynomial is the product of distinct linear factors over $\overline{F}$. On the other hand, over $\overline{F}$ the latter condition on the minimal polynomial is necessary and sufficient for the diagonalizability of A. Since the minimal polynomial of A is an invariant, depending only on the field containing the elements of A, it follows that A is diagonalizable over $\overline{F}$ if and only if A is completely reducible over F.

A set of $n \times n$ matrices with elements in F is called an algebra over F if it is a vector space over F (necessarily finite-dimensional) and is closed under matrix multiplication. The algebra over F generated by A is the smallest algebra of $n \times n$ matrices over F which contains A and I. This algebra consists of linear combinations over F of powers of A (including $I = A^0$) and a basis for the vector space underlying the algebra is $I, A, \cdots, A^{r-1}$ when A is of degree r.

A set $\mathfrak{M}$ of $n \times n$ matrices over F may be regarded as a set of linear endomorphisms of an n-dimensional vector space V over F with a fixed basis. A subspace W of V is said to be invariant under $\mathfrak{M}$ if each endomorphism in $\mathfrak{M}$ maps W into itself. An invariant subspace is said to be simple under $\mathfrak{M}$ if it contains no other invariant subspace except (0).

An algebra $\mathfrak{A}$ over F of $n \times n$ matrices containing I is said to be semi-simple if V is the direct sum of simple subspaces invariant under $\mathfrak{A}$. If $\mathfrak{A}$ is generated by A then it is known that $\mathfrak{A}$ is semi-simple if and only if A is completely reducible over F.

A theorem in the theory of algebras states that an algebra $\mathfrak{A}$ of matrices over a field F of characteristic zero is semi-simple if and only if the function $S(XY)$ (where $X, Y \in \mathfrak{A}$ and S denotes the trace function) is non-degenerate, *i.e.*, if $X \in \mathfrak{A}$ is such that $S(XY) = 0$ for every $Y \in \mathfrak{A}$, then $X = 0$.

Combining these results we see that if F is of characteristic zero and $\mathfrak{A}$ is the algebra over F generated by A, then the following are equivalent:

(1) A is diagonalizable over $\overline{F}$;
(2) the minimal polynomial of A is the product of distinct factors each irreducible over F;
(3) A is completely reducible over F;
(4) $\mathfrak{A}$ is semi-simple;
(5) $S(XY) = 0$ for fixed $X \in \mathfrak{A}$ and all $Y \in \mathfrak{A}$ implies $X = 0$.

Now let $x_0, \cdots, x_{r-1}$ be indeterminates and let $X = \sum_{i=0}^{r-1} x_i A^i$. Then

$$S(XA^j) = S\left(\sum_{i=0}^{r-1} x_i A^{i+j}\right) = \sum_{i=0}^{r-1} S(A^{i+j}) x_i, \quad (0 \leqq j \leqq r-1).$$

It is clear that the function $S(XY)$ will be degenerate on $\mathfrak{A}$ if and only if there exist elements $\alpha_0, \cdots, \alpha_{r-1} \in F$, not all zero, such that

$$\sum_{i=0}^{r-1} S(A^{i+j}) \alpha_i = 0, \qquad (0 \leqq j \leqq r-1).$$

Thus the non-degeneracy of $S(XY)$ is equivalent to the requirement that there exists only the trivial solution for a set of r homogeneous linear equations over F in r unknowns. This requirement is satisfied if and only if the determinant of coefficients, $\det [S(A^{i+j})]$, is non-vanishing.

MATHEMATICAL NOTES

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NOTES ON MATRIX THEORY—IV (AN INEQUALITY DUE TO BERGSTRÖM)

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1. Introduction. In a recent note,* Bergström proved the following interesting inequality:

Let A and B be positive definite matrices and let A_{ii}, B_{ii} denote the sub-matrices obtained by deleting the i-th row and column. Then

$$\frac{|A+B|}{|A_{ii}+B_{ii}|} \geqq \frac{|A|}{|A_{ii}|} + \frac{|B|}{|B_{ii}|}, \tag{1}$$

where $|\ |$ represents the determinant.

Bergström's proof is essentially a verification. We present two proofs below, the second of which lays bare the origin of the result.

2. First proof. The first proof is an immediate consequence of the result:

LEMMA 1: *If A is positive definite, then*

$$\phi(A) = \frac{|A|}{|A_{ii}|} = \min_{x} \sum_{i,j=1}^{N} a_{ij}x_ix_j, \tag{1}$$

where x is constrained by

$$x_i = 1. \tag{2}$$

From this it is clear that $\phi(A+B) \geqq \phi(A)+\phi(B)$.

We shall not present the proof, which is easily obtained by the use of a Lagrange multiplier, since Lemma 1 is a special case of the more general result established in the next section.

3. Second proof. We begin by establishing

LEMMA 2: *If A is positive definite, then*

$$(x, Ax)(y, A^{-1}y) \geqq (x, y)^2, \tag{1}$$

for all x and y.

Here (x, y) denotes the inner product of x and y and (x, Ax) the quadratic form $\sum_{i,j}a_{ij}x_ix_j$. A^{-1} is the inverse of A.

Proof of Lemma 2: Reduce A to diagonal form by an orthogonal matrix T,

* H. Bergström, "A triangle inequality for matrices," *Den 11te Skandinaviske Matematikerkongress, Trondheim, 1949*, Oslo, 1952, pp. 264–267.

i.e., $T'AT=L$, $T'=T^{-1}$. Let $x=Tu$, $y=Tv$. Then (1) becomes

$$\left(\sum_{i=1}^{N}\lambda_i u_i^2\right)\left(\sum_{i=1}^{N} v_i^2/\lambda_i\right) \geqq \sum_{i=1}^{N} u_i v_i, \tag{2}$$

which is the Cauchy-Schwarz inequality.

Since the inequality becomes an equality for suitable choice of x, we have

$$\min_x \frac{(x, Ax)}{(x, y)^2} = \frac{1}{(y, A^{-1}y)} = \psi(A). \tag{3}$$

From this it is immediate that

$$\psi(A+B) \geqq \psi(A)+\psi(B). \tag{4}$$

The case $y_i=1$, $y_j=0$, $j\neq i$ yields Bergström's result.

A REMARK ON MATRIC POLYNOMIALS AND SIMILARITY OF MATRICES

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Let F be a field and let F_n be the set of all n-rowed square matrices with elements in F. It is customary to call $A(x)=A_m x^m+\cdots+A_1x+A_0$ a *matric polynomial* if A_i is in F_n $(i=0, \cdots, m)$. If C is also in F_n, we define, as usual, $A_R(C)=A_mC^m+\cdots+A_1C+A_0$. Our purpose is to note a rather important consequence of the following innocent lemma.

Lemma. *If $B(x)$ is also a matric polynomial, then*

$$(A(x)B(x))_R(C) = A_mB_R(C)C^m+\cdots+A_1B_R(C)+A_0B_R(C).$$

Proof. This is clear, since $A(x)B(x)=A_mB(x)x^m+\cdots+A_1B(x)x+A_0B(x)$.

Corollary 1. *If $B_R(C)=0$, then $(A(x)B(x))_R(C)=0$.*

Our next Corollary is quite useful in certain parts of matric theory.

Corollary 2. *If M and N are in F_n and $P(x)(Ix-M)Q(x)=Ix-N$ for matric polynomials $P(x)$ and $Q(x)$ which have matric polynomial inverses, then $P=Q_R(N)$ is non-singular and $N=P^{-1}MP$.*

Proof. We have $(Ix-M)Q(x)=P^{-1}(x)(Ix-N)$ and it is a consequence of Corollary 1 that $((Ix-M)Q(x))_R(N)=0$. Apply the Lemma to obtain $Q_R(N)N-MQ_R(N)=0$, or $PN=MP$. Now $Q^{-1}(x)Q(x)=I$, and let us write $Q^{-1}(x)=U_tx^t+\cdots+U_1x+U_0$. Apply the Lemma again, this time to $I=(Q^{-1}(x)Q(x))_R(N)$ to obtain

$$I=U_tQ_R(N)N^t+\cdots+U_1Q_R(N)N+U_0Q_R(N)=U_tPN^t+\cdots+U_1PN+U_0P.$$

Now use $PN=MP$ repeatedly to find that

$$I=U_tM^tP+\cdots+U_1MP+U_0P=(Q_R^{-1}(M))P.$$

This proves that P is non-singular and $N=P^{-1}MP$ follows from $PN=MP$.

Our proof of Corollary 2 may be put to use in the actual *computation* of the matrix P which yields canonical forms for A in F_n under similarity. Thus we may reduce $Ix-A$ to Smith canonical form $S(x)$ by elementary operations. We carefully record the *column* operations used, being as economical as we can in our use of such operations. Let J be the Jordan canonical matrix* and let K be the classical canonical matrix* of A. It is easy to list elementary operations which replace $S(x)$ by $Ix-J$. We then have $P(x)(Ix-A)Q(x)=Ix-J$ and hence also $J=P^{-1}AP$ with $P=Q_R(J)$, $P^{-1}=Q^{-1}(A)$. To obtain $Q^{-1}(x)$ we apply to I the elementary column operations which led to $Q(x)$ *in reverse order*. One may carry out an analogous procedure to obtain P_1 (and P_1^{-1}) such that $P_1^{-1}AP_1=K$.

Our proof of Corollary 2 and our remarks concerning computation are valid if F is a division ring. It is not difficult to adapt our proof to the more general case of σ-similarity of matrices ($N=P^{-1}MP^{\sigma}$, σ an automorphism of F) which is appropriate to a discussion of the matrices which represent a *semi-linear* transformation T with automorphism σ. For simplicity we have chosen to write our proof for the case σ the identity automorphism, T a linear transformation.

Although we believe that our *proof* of Corollary 2 is new, we should like to note in conclusion that all of the results we have stated for F a *field* may be found in many of the large number of recent texts on matric theory. For the case of a division ring, the reader should consult the book *The Theory of Rings* by N. Jacobson.

* We follow N. Jacobson, *Lectures in Abstract Algebra II, Linear Algebra*, pp. 93–94, in our use of these terms.

PARTICULAR INTEGRALS OF LINEAR DIFFERENTIAL EQUATIONS

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In this note a convenient method is developed for obtaining a particular solution of the linear differential equation with constant coefficients

$$(D^n + a_1D^{n-1} + \cdots + a_n)Y = f(x). \tag{1}$$

The method is simpler than the Lagrange method of "variation of parameters," especially when the auxiliary equation has repeated roots. Also, the procedure is more easily extended to the case of simultaneous equations than is Lagrange's method.

We start by assuming a solution of the form

$$Y = \int_0^x G(x-\xi)f(\xi)d\xi, \tag{2}$$

where the function G is to be determined.† Then

† The reader who is familiar with operational methods will recognize a well known theorem on convolution integrals. This type of solution is also suggested by the formula resulting from Lagrange's method when the auxiliary equation has distinct roots.

$$Y' = G(0)f(x) + \int_0^x G'(x-\xi)f(\xi)d\xi. \tag{3}$$

If now it is specified that $G(0)$ be zero, then

$$Y' = \int_0^x G'(x-\xi)f(\xi)d\xi. \tag{4}$$

Similarly,

$$Y'' = G'(0)f(x) + \int_0^x G''(x-\xi)f(\xi)d\xi, \tag{5}$$

and if we take $G'(0)=0$ this reduces to

$$Y'' = \int_0^x G''(x-\xi)f(\xi)d\xi. \tag{6}$$

Proceeding in this manner, we find that

$$Y^{(n)} = G^{(n-1)}(0)f(x) + \int_0^x G^{(n)}(x-\xi)f(\xi)d\xi, \tag{7}$$

where the condition $G^{(k)}(0)=0$ has been applied to G and its first $(n-2)$ derivatives. Substitution into (1) yields

$$G^{(n-1)}(0)f(x)+\int_0^x [G^{(n)}(x-\xi)+a_1G^{(n-1)}(x-\xi)+\cdots+a_nG(x-\xi)]f(\xi)d\xi=f(x), \tag{8}$$

and it is seen that (5) is a solution of (7) provided $G(s)$ is a solution of the homogeneous equation, and

$$G^{(n-1)}(0) = 1.$$

Thus G is completely determined by the homogeneous equation and the initial conditions $G(0)=G^{(k)}(0)=0$, $k=1, \cdots, (n-2)$, $G^{(n-1)}(0)=1$.

As an example we consider the equation

$$(D^4 - 2a^2D^2 + a^4)Y = f(x). \tag{9}$$

The solution to the homogeneous equation which satisfies the conditions $Y(0)=Y'(0)=Y''(0)=0$, $Y'''(0)=1$, is

$$\frac{1}{2a^3}[(ax)\cosh(ax) - \sinh(ax)]. \tag{10}$$

Hence a particular solution of (17) is

$$\frac{1}{2a^3}\int_0^x [a(x-\xi)\cosh a(x-\xi) - \sinh a(x-\xi)]f(\xi)d\xi. \tag{11}$$

The present method is also advantageous in the case of simultaneous equations for which the method of variation of parameters is not directly applicable. For example, consider the pair of equations:

$$(12)\qquad \begin{aligned} &[a_0D^n + a_1D^{n-1} + \cdots + a_n]Y + [b_0D^n + b_1D^{n-1} + \cdots + b_n]Z = f(x),\\ &[c_0D^n + c_1D^{n-1} + \cdots + c_n]Y + [d_0D^n + d_1D^{n-1} + \cdots + d_n]Z = g(x), \end{aligned}$$

where the a's, b's, c's and d's are constants. A particular solution of this system can be obtained by taking

$$(13)\qquad \begin{aligned} Y &= \int_0^x G_1(x-\xi)f(\xi)d\xi + \int_0^x G_2(x-\xi)g(\xi)d\xi,\\ Z &= \int_0^x H_1(x-\xi)f(\xi)d\xi + \int_0^x H_2(x-\xi)g(\xi)d\xi. \end{aligned}$$

Substituting into the system, we find that G_1 and H_1, G_2 and H_2 are solutions of the homogeneous equations subject to the conditions

$$(14)\qquad \begin{aligned} &\begin{cases} G_1(0) = G_1^{(k)}(0) = 0 \\ H_1(0) = H_1^{(k)}(0) = 0, \ [k = 0, 1, \cdots, (n-2)] \end{cases} \qquad \begin{cases} a_0G_1^{(n-1)}(0) + b_0H_1^{(n-1)}(0) = 1 \\ c_0G_1^{(n-1)}(0) + d_0H_1^{(n-1)}(0) = 0 \end{cases}\\ &\begin{cases} G_2(0) = G_2^{(k)}(0) = 0 \\ H_2(0) = H_2^{(k)}(0) = 0, \ [k = 0, 1, \cdots, (n-2)] \end{cases} \qquad \begin{cases} a_0G_2^{(n-1)}(0) + b_0H_2^{(n-1)}(0) = 0 \\ c_0G_2^{(n-1)}(0) + d_0H_2^{(n-1)}(0) = 1. \end{cases} \end{aligned}$$

It should be noted that the determinant $[a_0d_0 - b_0c_0]$ can be assumed to be non-vanishing, for otherwise it would be possible to reduce the order of the system by one through suitable linear combinations of the equations.

As an example, consider the system

$$(15)\qquad \begin{aligned} &[D^2 - a^2]Y - 2aDZ = f(x),\\ &2aDY + [D^2 - a^2]Z = g(x). \end{aligned}$$

The solution of the homogeneous equations which satisfies the conditions $Y(0) = Z(0) = 0$, $Y'(0) = 1$, $Z'(0) = 0$ is

$$(16)\qquad Y = x\cos(ax), \qquad Z = -x\sin(ax),$$

and the solution satisfying the conditions $Y(0) = Z(0) = 0$, $Y'(0) = 0$, $Z'(0) = 1$ is

$$(17)\qquad Y = x\sin(ax), \qquad Z = x\cos(ax).$$

Therefore a solution to the non-homogeneous equations is

$$(18)\qquad \begin{aligned} Y &= \int_0^x (x-\xi)\cos a(x-\xi)f(\xi)d\xi + \int_0^x (x-\xi)\sin a(x-\xi)g(\xi)d\xi\\ Z &= -\int_0^x (x-\xi)\sin a(x-\xi)f(\xi)d\xi + \int_0^x (x-\xi)\cos a(x-\xi)g(\xi)d\xi. \end{aligned}$$

CLASSROOM NOTES

EDITED BY G. B. THOMAS, Massachusetts Institute of Technology

All material for this department should be sent to G. B. Thomas, Department of Mathematics, Massachusetts Institute of Technology, Cambridge 39, Mass.

EQUATIONS REDUCIBLE TO LINEAR FORM

E. A. NORDHAUS, Michigan State College

In standard texts on ordinary differential equations the general solution of Bernoulli's equation $dy/dx = yA(x) + y^n B(x)$, where $n \neq 0, 1$, is obtained by reducing it to linear form using the transformation $z = y^{1-n}$. This artifice is usually unmotivated and although the observations made in this note may possibly be embedded somewhere in the extensive literature, it is apparently not widely known that there is a large class of first order differential equations which may be reduced to linear form by a transformation of the dependent variable.

It will be shown (under suitable differentiability assumptions for the functions involved) that an equation $dy/dx = f(x, y)$ can be reduced to linear form if and only if the right member has the form $f(x, y) = A(x)M(y) + B(x)N(y)$, where $MN' - NM' = cN$, c a non-zero constant. An effective transformation is $z = M/N$. Throughout this note accents are used to indicate derivatives taken with respect to the argument of the function involved. In the case of the Bernoulli equation we may choose $M(y) = y$, $N(y) = y^n$ and verify that $MN' - NM' = (n-1)N$, $z = M/N = y^{1-n}$. Additional examples may be readily constructed by selecting $N(y)$ and solving the resulting linear equation $dM/dy = MN'/N - c$. A particular solution is $M(y) = N\int dy/N$.

The proof is immediate.* We note that an equation (1) $y' = f(x, y)$ can be put into the linear form (2), $z' = a(x)z + b(x)$, by a transformation (3), $z = q(y)$, if and only if (2) can be put into form (1) by (3). This implies that $q'(y)y' = a(x)q(y) + b(x)$, so $f(x, y)$ must have the form $A(x)M(y) + B(x)N(y)$ with $M = kq/q'$ and $N = h/q'$. Then $q = (h/k) \cdot (M/N)$, and since

$$q' = \frac{h}{k} \cdot \frac{NM' - MN'}{N^2} = \frac{h}{N},$$

it follows that $NM' - MN' = kN$.

The method used above may be readily extended to the reduction of equations of higher order to linear form, and similar necessary and sufficient conditions are obtainable. However, as might be expected, the form of the function f grows increasingly specialized, so that these extensions seem to be of little interest.

* The author wishes to thank the referee for suggesting the following simplified proof.

MEAN VALUE THEOREMS FOR FUNCTIONS WITH LEFT AND RIGHT DERIVATIVES

W. R. Utz, University of Missouri

This note is concerned with the following generalizations of Rolle's theorem and two well-known mean value theorems. Throughout the note $f'_-(x)$ and $f'_+(x)$ denote left and right derivatives, respectively.

THEOREM 1. *If $f(x)$ is a continuous function for $a \leqq x \leqq b$, $f(a) = f(b) = 0$ and $f'_-(x)$, $f'_+(x)$ exist in $a < x < b$, then there exist numbers ξ, p, q with $a < \xi < b$, $p \geqq 0$, $q \geqq 0$, $p + q = 1$ for which*

$$pf'_+(\xi) + qf'_-(\xi) = 0.$$

THEOREM 2. *If $f(x)$ is a continuous function for $a \leqq x \leqq b$, $f'_+(x)$ and $f'_-(x)$ exist for $a < x < b$, then there exist numbers ξ, p, q with $a < \xi < b$, $p \geqq 0$, $q \geqq 0$, $p + q = 1$ for which*

$$\frac{f(b) - f(a)}{b - a} = pf'_+(\xi) + qf'_-(\xi).$$

THEOREM 3. *If $f(x)$ and $g(x)$ are continuous for $a \leqq x \leqq b$ and $f'_+(x)$, $f'_-(x)$, $g'_+(x)$, $g'_-(x)$ exist on $a < x < b$, then there exist numbers ξ, p, q with $a < \xi < b$, $p \geqq 0$, $q \geqq 0$, $p + q = 1$ for which*

$$[pf'_+(\xi) + qf'_-(\xi)](g(b) - g(a)) = [pg'_+(\xi) + qg'_-(\xi)](f(b) - f(a)).$$

Theorems 1 and 2 are given by Jovan Karamata [*Sur la formule des accroissements finis*, Srpska Akad. Nauka. Zbornik Radova 7, Matematički Inst., vol. 1, 1951, pp. 119–124 (Serbo-Croatian. French summary.)] and Theorem 3 is given by V. Vučković [*Quelques extensions des théorèmes de moyenne*, Srpska Akad. Nauka. Zbornik Radova, vol. 18, Matematički Inst. 2, 1952, pp. 159–166 (Serbo-Croatian. French summary.)]. However, in both papers the authors maintain that a stronger conclusion is secured, *viz.*, "$p \geqq 0$, $q \geqq 0$" is, in each theorem, replaced by "$p > 0$, $q > 0$." To see that one can not secure this conclusion in any one of the three theorems let $a = -1$, $b = 1$,

$$f(x) = \begin{cases} x^3 - 1, & x \geqq 0; \\ -x - 1, & x < 0. \end{cases}$$

In Theorems 1 and 2, q must be 0 and if we take $g(x) = x$, then q must be 0 in Theorem 3.

The author has found the theorems suitable exercises for advanced calculus classes. The proofs of Theorems 2 and 3 follow from Theorem 1 by arguments parallel to those used to secure the corresponding theorems for differentiable functions from Rolle's theorem.

We shall sketch a proof of Theorem 1. If $f(x) \equiv 0$ one may take, as usual,

$\xi=(b-a)/2$, $p=q=1/2$. If $f(x)\not\equiv 0$ then $f(x)$, being continuous, assumes a positive maximum or negative minimum (or both) in $a<x<b$. Let $x=\xi$, $a<\xi<b$, be such that $f(\xi)>0$ is a maximum or $f(\xi)<0$ is a minimum. If $f'_-(\xi)=f'_+(\xi)=f'(\xi)$, one continues as in Rolle's theorem with $p=q=1/2$. Suppose $f'_-(\xi)\neq f'_+(\xi)$. If $f'_-(\xi)=0$, let $p=0$, $q=1$; if $f'_+(\xi)=0$, let $p=1$, $q=0$. If $f'_-(\xi)f'_+(\xi)\neq 0$, then this product is negative as can be seen from the definitions of $f'_-(x)$ and $f'_+(x)$. In this final case the equations $p+q=1$ and $pf'_+(\xi)+qf'_-(\xi)=0$ are satisfied by

$$p=\frac{f'_-(\xi)}{f'_-(\xi)-f'_+(\xi)}\geqq 0, \qquad q=\frac{f'_+(\xi)}{f'_+(\xi)-f'_-(\xi)}\geqq 0.$$

ON EQUIVALENCE RELATIONS

W. T. Guy, Jr., The University of Texas

Probably many teachers who teach both elementary and advanced courses are frequently struck with the idea that some of the concepts and troublesome spots in elementary courses could be cleared up much more quickly, as well as more easily, with the "machinery" of the advanced courses. Despite the fact that usually this more powerful structure can not be brought to bear on the elementary work, it is felt that we should actively look for cases in which it can be used with profit for our students. The purpose of this note is to set forth another plea that the concept of equivalence relation be introduced early in the students' training—even as early as in the algebra course of our present day high schools.

College and university teachers of my acquaintance frequently complain that the students do not know or appreciate the difference between an identity, a definition, and a conditional equation. I have accused college algebra students of using the equal sign as the "pause that refreshes" when they come to a place where they hesitate or have to stop to think. After deciding what to do they start writing just to the right of this "pause" sign, although they did not intend to do so originally. Many of us have received papers from students containing symbols with equal signs sprinkled around like salt and pepper, similar to

$$y^2+4y=-4=(y+2)^2=0=y+2=0=y=-2, \text{ or}$$
$$\sum 1/n^2=\sum 1/n^p=p>1=\text{converges by the } p\text{-series test.}$$

Needless to say, such scribbling shows a complete lack of understanding of or appreciation for an equal sign.

Recently a graduate student who was supposed to be well along in his work said that the usual hypotheses for an equivalence relation were redundant. He referred to the

Definition. *An equivalence relation (symbol* $\sim$*) is said to be defined over a*

non-empty set S of elements if and only if given any two elements a and b of S either $a\sim b$ (read a equivalent to b) or $(a\sim b)'$ (read a not equivalent to b), but not both, such that

(i) $a\sim a$ *for every a of S,*

(ii) $a\sim b$ *implies* $b\sim a$ *for every a and b of S, and*

(iii) $a\sim b$ *and* $b\sim c$ *imply* $a\sim c$ *for every a, b, and c of S.*

Freshmen students quickly see that our usual use of an equal sign is a special instance of an equivalence relation defined over the set of objects under discussion. They also realize that this definition does not define an equal sign.

The student cited above was disturbed by the hypothesis $a\sim a$ and expressed surprise when sets with relations were exhibited such that any two of the three hypotheses were fulfilled whereas the remaining one was not. Such examples can be constructed without end and three have been selected to be displayed here. The hypotheses are satisfied vacuously in one of then. This was done deliberately since it has been my experience that such examples will stimulate quite a heated class discussion, after which the students will be able to construct more self-satisfying examples. It is felt that after such discussions and examples the students involved will have a much better appreciation for the equal symbol as it is is usually used.

Example 1. Let the elements of the set S_1 be denoted and arranged in a vertical column as indicated. By $x\sim y$, for any x and y of S_1, we mean that x and y are distinct and that x and y are in the same horizontal row. By $(x\sim y)'$ we mean that $x\sim y$ is false.

$$S_1: \quad a$$
$$b$$
$$c$$

Here hypothesis (i) is not fulfilled whereas (ii) and (iii) are.

Example 2. Let the elements of the set S_2 be denoted and arranged as indicated. By $x\sim y$, for any x and y of S_2, we mean that x and y denote the same element or that y is to the "right of" x. By $(x\sim y)'$ we mean that $x\sim y$ is false.

$$S_2: \quad a, b$$
$$c$$
$$d$$

Here hypothesis (ii) is not fulfilled whereas the others are.

Example 3. Let the elements of S_3 be denoted and arranged as indicated. By $x\sim y$, for any x and y of S_3, we mean that x and y denote the same element or that y is "next to" x. By $(x\sim y)'$ we mean that $x\sim y$ is false.

$$S_3: \quad a, b, c, d$$

Here hypothesis (iii) is not fulfilled whereas the others are.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1156. *Proposed by W. A. Hockings, Calumet, Michigan*

The conversation of problem E 1126 continues:

Jones: "Both Brown and I also have ranches in Todd County. They are oblong in shape and a whole number of miles in length and width."

Brown: "Yes, and your ranch is nine times as large as mine."

Smith: "I know the length of Jones' ranch, but I can't figure the area."

Green: "I know the area of Jones' ranch, but I can't determine the length without more information. Is the width less than half the length?"

Smith: "Before you answer, Jones, let me see if I can now determine the area." He deliberates for a minute. "No, I still do not know the area."

Jones answers the question.

Green: "I now know the length."

Smith: "Although I did not hear the answer to Green's question, I now know the area."

What are the dimensions of Jones' ranch in miles?

E 1157. *Proposed by T. F. Mulcrone, St. Charles College, Grand Coteau, La.*

Determine the smallest collection of positive integers such that for any integer k, $0<k\leqq c$, where c is a fixed integer, there is a subcollection whose sum is k.

E 1158. *Proposed by J. E. Hanson, Johns Hopkins University*

Find the roots of

$$\sum_{i=0}^{n}\binom{n+i+1}{2i+1}x^i=0,\ n\geqq 1.$$

E 1159. *Proposed by Albert Newhouse, University of Houston*

In finding the greatest common divisor (a, b) of a and b (either integers or polynomials) by the Euclidean algorithm there appear, in the successive steps, quotients q_i and remainders r_i. Show that $(a, b)=as+bt$, where

$$s = (-1)^{k-1}\begin{vmatrix} q_2 & 1 & 0 \cdots 0 \\ -1 & q_3 & 1 \cdots 0 \\ 0 & -1 & q_4 \cdots 0 \\ \cdot & \cdot & \cdot \cdots \cdot \\ 0 & 0 & 0 \cdots q_k \end{vmatrix}, \quad t = (-1)^k \begin{vmatrix} q_1 & 1 & 0 \cdots 0 \\ -1 & q_2 & 1 \cdots 0 \\ 0 & -1 & q_3 \cdots 0 \\ \cdot & \cdot & \cdot \cdots \cdot \\ 0 & 0 & 0 \cdots q_k \end{vmatrix}.$$

E 1160. *Proposed by Hüseyin Demir, Zonguldak, Turkey*

Prove that in a complete quadrilateral the isotomic line of any side with respect to the triangle formed by the other three is parallel to the Newton line of the quadrilateral.

SOLUTIONS

Sum of Cubes as Difference of Two Squares

E 1108 [1954, 194 and 643]. *Proposed by Renê Bloch, Humanistisches Gymnasium, Basle, Switzerland*

Let n, k, $a+1$ be three positive integers which are not all odd. Express $\Sigma_{i=0}^{k}(n+ai)^3$ as a difference of two integer squares.

II. *Comment by W. B. Carver, Cornell University.* This problem, and the proposer's solution [1954, 644], might lead one to suppose that when n, k, and $a+1$ are all odd the sum of the cubes *cannot* be expressed as the difference of two integer squares. This is not the case, as, for instance, when $n=5$, $k=3$, $a+1=5$, the sum of the four cubes is $7964=1992^2-1990^2$. It might be of some interest to ask what are necessary and sufficient conditions on the positive integers n, k, and a such that the sum of cubes

$$S = \sum_{i=0}^{k} (n + ai)^3$$

cannot be expressed as a difference of two squares. The answer is that we must have

$$n \text{ odd}, \quad \left.\begin{matrix} k \\ a+1 \end{matrix}\right\} \equiv 1 \bmod 4.$$

Any integer S, except when $S \equiv 2 \bmod 4$, may be *very simply* expressed as the difference of two squares thus

$$S = [(S+1)/2]^2 - [(S-1)/2]^2 \quad \text{when} \quad S \text{ is odd},$$
$$S = [S/4+1]^2 - [S/4-1]^2 \quad \text{when} \quad S \equiv 0 \bmod 4.$$

In the remaining case, $S \equiv 2 \bmod 4$, S cannot be the difference of two squares, and the sum of the cubes has this form if and only if n, k, and a satisfy the conditions given above.

The Length of Smith's Ranch

E 1126 [1954, 470]. *Proposed by J. V. Pennington, Houston, Texas*

Smith: "Down in Todd County, which is a 23-mile square, I have a ranch—oblong in shape—measuring a whole number of miles each way."

Jones: "Hold on a minute. I happen to know the area of your ranch; let me see if I can figure its length." He figures furiously. "I need more information. Is the width more than half the length?"

Smith answers the question.

Jones: "I now know the length."

Brown: "I too know the area and, although I did not hear your answer to Jones' question, I too can tell you the length."

Green: "I did not know the area, but I can now tell you the length."

How many miles long is the ranch?

Solution by Frank Kocher, Pennsylvania State University. Assume the sides of the ranch are parallel to the sides of the county. Now, the area of the ranch has more than one factorization, and in at least one factorization $2W<L$ and in at least one factorization $2W>L$. Had there been only one factorization of each type Brown would have needed the answer to Jones' question. Hence, the area must have at least three factorizations with either only one of them of the type $2W<L$ or else only one of them of the type $2W>L$. Consideration of the multiplication table up to 22×23 shows a number of cases of integers with three factorizations, none with more. If the width were more than half the length no unique length would be determined, but if the width is less than half the length, the length is 20 miles. The area is either 180 or 120 sq. mi.

It is interesting to note that if Jones, Brown, Green, and we do not assume the sides of the ranch parallel to the sides of the county, then the area also is uniquely determined, and is 180 sq. mi., for a 5×24 oblong may fit into a 23×23 square.

Also solved by P. M. Anselone, J. L. Botsford, Julian Braun, P. L. Chessin, W. A. Hockings, A. R. Hyde, M. S. Klamkin, Sidney Kravitz, Octave Levenspiel, D. C. B. Marsh, C. S. Ogilvy, Walter Penney, L. A. Ringenberg, Arnold Walfisz, and the proposer.

Number of Squares on an $m\times n$ Checker Board

E 1127 [1954, 470]. *Proposed by Edmund DeWan, Harpur College*

Enumerate the total number of squares appearing on an $m\times n$ checker board, $m\geqq n$. Solve the analogous problem for rectangular parallelopipeds.

Solution by Julian Braun, Aberdeen Proving Ground, Maryland. The number of squares measuring i units on a side (the side of a smallest square being taken as the unit) is $(m+1-i)(n+1-i)$ and $i_{\max}=n$, whence the total number of squares, T_2, is given by

$$T_2 = \sum_{i=1}^{n} (m+1-i)(n+1-i) = \sum_{i=0}^{n} (m-i)(n-i)$$
$$= n(n+1)(3m+1-n)/6.$$

Similarly, the number of cubes, T_3, in a rectangular parallelopiped subdivided into an $m \times n \times p$ "checkerblock," $m \geqq p$, $n \geqq p$, is

$$T_3 = \sum_{i=1}^{p} (m+1-i)(n+1-i)(p+1-i) = \sum_{i=0}^{p} (m-i)(n-i)(p-i)$$
$$= p(p+1)[6mn - (p-1)(2m+2n-p)]/12.$$

It may also be asked how many rectangles and rectangular parallelopipeds appear in the respective manifolds. The former is given by

$$\sum_{i=0}^{m} (m-i) \sum_{j=0}^{n} (n-j) = mn(m+1)(n+1)/4$$

and the latter by

$$\sum_{i=0}^{m} (m-i) \sum_{j=0}^{n} (n-j) \sum_{k=0}^{p} (p-k) = mnp(m+1)(n+1)(p+1)/8.$$

The extension to an N dimensional manifold is evident.

Also solved by P. M. Anselone, A. P. Boblétt, J. E. Freund, A. R. Hyde, M. S. Klamkin, D. C. B. Marsh, Leo Moser, T. F. Mulcrone, Walter Penney, L. A. Ringenberg, Azriel Rosenfeld, Milton Scharf, R. E. Shafer, C. W. Trigg, Alan Wayne, and the proposer.

Geometrical Solution of an Old Calculus Problem

E 1128 [1954, 470]. *Proposed by C. S. Ogilvy, Hamilton College*

Show that the point P at eye level from which a picture high on a vertical wall subtends the maximum angle can be found as follows. With eye-level line as directrix and bottom of picture as focus, draw a parabola; from the intersection of this parabola with a line through the center of the picture parallel to the directrix, a line perpendicular to the directrix meets it at P.

Solution by Leon Bankoff, Los Angeles, Calif. It is clear that a circle (O) can be drawn through the top T and the bottom B of the picture, and tangent to the directrix at P. It follows that O is equidistant from P, T, and B, and is the point of concurrence of

a) the perpendicular bisector of TB,

b) the perpendicular to the eye-level line at P,

c) the parabola with focus at B or T and with the eye-level line as directrix,

d) the ellipse with foci B and T and with semi-major axis equal to OP.

Also solved by Norman Anning, J. E. Darraugh, A. L. Epstein, Herta

Freitag, B. K. Gold, A. R. Hyde, J. M. Kingston, J. D. E. Konhauser, D. C. B. Marsh, T. F. Mulcrone, M. J. Pascual, Walter Penney, L. A. Ringenberg, Azriel Rosenfeld, S. H. Sesskin, Sister M. Stephanie, D. R. Sudborough, J. A. Tierney, C. W. Trigg, Chih-yi Wang, Roscoe Woods, David Zeitlin, and the proposer.

Editorial Note. The above treatment applies equally well to the case of a slanted picture.

A Power-free Arithmetic Progression

E 1129 [1954, 470]. *Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn*

Find an integral arithmetic progression with an arbitrarily large number of terms such that no term is a perfect rth power for $r=2, 3, \cdots, N$. Is this still possible if $N=\infty$?

Solution by Azriel Rosenfeld, Columbia University. The progression 2, 6, 10, $\cdots, 4k+2, \cdots$ can contain no perfect powers whatsoever. For, a power of an odd integer is odd, and a power of an even integer must be divisible by 4.

Also solved by W. E. Briggs, A. R. Hyde, Leo Moser, J. B. Muskat, and R. E. Shafer.

An obvious solution, pointed out by Hyde, is any arithmetic progression with common difference $d=0$ and with the (invariant) term chosen so as not to be an integral power of an integer, for example the progression 3, 3, 3, $\cdots$, Hyde, Moser, and Shafer offered deeper solutions to the problem.

A Criterion for a Triangle to be Isosceles

E 1130 [1954, 470]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Let the perpendicular bisector of the median BB' of triangle ABC and the tangent at B to the circumcircle of triangle ABC cut the line AC in points M and N respectively. Show that triangle ABC is isosceles with vertex at A if and only if $AM/AN=3/4$.

I. *Solution by Chih-yi Wang, University of Minnesota.* Let the coordinates of A, B, C be $(-a, 0)$, $(0, b)$, $(c, 0)$ respectively. Then the coordinates of M and N can be easily found as, respectively,

$$\left(\frac{b^2}{a-c}-\frac{a-c}{4}, 0\right) \text{ and } \left(\frac{b^2+ac}{a-c}, 0\right).$$

Let $AM/AN=k$. Then we have, after simplification,

$$(1) \qquad (c+a)-\sqrt{a^2+b^2}=(\sqrt{4-4k}-1)\sqrt{a^2+b^2}.$$

The left side of (1) is equal to 0 if and only if $k=3/4$.

II. *Solution by Roscoe Woods, State University of Iowa.* It is obvious that the points M and N are not finite unless $a\neq c$. Let P denote the midpoint of the

median BB' and let ϕ denote the acute angle between the median BB' and the side AC. From the right triangle $B'PM$, it follows that $B'M=BB'/(2\cos\phi)$. By substituting the value of $\cos\phi$ from the triangle $B'BC$ and making use of the fact that the square of the length of the median BB' is $(2a^2+2c^2-b^2)/4$ and that $AM=b/2+B'M$, a short reduction shows that $AM=b(4c^2-b^2)/(4c^2-4a^2)$. Since $AN(AN-b)=NB^2=c^2+AN^2-2c\,AN\cos A$, we find that, upon replacing $\cos A$ by its value $(b^2+c^2-a^2)/2bc$ and collecting terms, $AN=bc^2/(c^2-a^2)$. It now follows that $AM/AN=(4c^2-b^2)/4c^2$. The solution is completed by noting that this ratio is $3/4$ if and only if $b=c$.

Also solved by J. L. Botsford, A. R. Hyde, Josef Langr, Beckham Martin, Margaret Olmsted, L. A. Ringenberg, and the proposer.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4617 [1954, 719]. *Correction.*

In place of the condition $x^2+y\leqq 1$, read $x^2+y^2\leqq 1$.

4628. *Proposed by L. Carlitz, Duke University*

Let p be an odd prime and let $P_m(x)$ denote the Legendre polynomial of degree m, where $p=2m+1$. Show that

$$P_m(3)\equiv P_m(-3)\equiv\begin{cases}0 \pmod p & (m \text{ odd})\\ 2a \pmod p & (m \text{ even}),\end{cases}$$

where a is the unique odd integer determined by

$$p=a^2+b^2,\qquad a\equiv b+1 \pmod 4.$$

4629. *Proposed by M. Henriksen and S. Perlis, Purdue University*

A commutative ring R is called a radical ring (in the sense of Jacobson) if every element has a quasi-inverse; *i.e.*, for every a in R, there is an x in R such that $a+x+ax=0$. In particular, if every element of R is nilpotent, then R is a radical ring. Need every subring of a radical ring be a radical ring?

4630. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Having given two confocal conics C_1 and C_2, let M_1 on C_1 and M_2 on C_2 be so chosen that the tangents at these points are perpendicular. Show that the envelope of M_1M_2 is another conic having the same foci and having asymptotes passing through the intersections of C_1 and C_2.

4631. *Proposed by F. H. Northover, Memorial University of Newfoundland*

Evaluate the unending continued fraction

$$1+\frac{1}{2+}\,\frac{1}{3+}\,\frac{1}{4+}\cdots.$$

4632. *Proposed by H. W. Becker, Station WOW, Omaha, Nebraska*

As quoted in L. E. Dickson, *History of the Theory of Numbers*, vol. II, p. 504, "C. Gill treated the problem to find n squares the sum of any $n-1$ of which is a square. He gave his solution for $n=5$ and remarked that the smallest numbers given by his formulas are so very large as to discourage any attempt to compute them." Perhaps allowing duplications among the n values, find a computable formula for general n, or for a particular $n\geqq 5$.

SOLUTIONS

Binomial Coefficients

4566 [1953, 716]. *Proposed by C. C. Chevalley, Columbia University, and Paul Erdös, University of Notre Dame*

Show that the following relations are impossible for $n>2$:

$$\binom{2n}{n}=2\binom{2n}{k},\qquad \binom{2n+1}{n}=2\binom{2n+1}{k}.$$

Solution by Ernst Trost, Technikum Winterthur, Zürich, Switzerland. We consider first the relation

$$\frac{(2n)!}{n!n!}=\binom{2n}{n}=2\binom{2n}{k}=2\,\frac{(2n)!}{k!(2n-k)!},\qquad n>1, \tag{1}$$

where, of course, we can suppose $0<k<n$. From (1) follows

$$\frac{2n-k}{k+1}\cdot\frac{2n-k-1}{k+2}\cdots\frac{n+2}{n-1}\cdot\frac{n+1}{n}=2. \tag{2}$$

Equation (2) is impossible for $k+1=n>1$, hence $n-k>1$. Because each of the $n-k$ factors on the left is >1 we may assume that $2n-k<2(k+1)$, whence $k+1\geqq 2(n-k)>2$. Now we make use of the following theorem of Sylvester:* If

* For a proof see P. Erdös, A theorem of Sylvester and Schur, *Journal Lond. Math. Soc.*, vol. 9, 1934, pp. 282–288.

$r>s$, then in the set of s consecutive integers $r, r+1, r+2, \cdots, r+s-1$ there is a number having a prime divisor greater than s. Hence we infer that there is a prime $p \geqq 2(n-k)>2$ dividing one of the integers $k+1, \cdots, 2n-k-1$. If $k+1 \equiv 0 \pmod{p}$ the next integer having the prime divisor p is $k+1+p \geqq 2n-k+1$, which implies that in (2) a factor $p^e (e \geqq 1)$ occurs only once. If p^e occurs in the denominator of (2) the quotient cannot be integral, if p^e occurs in the numerator an integral quotient must be of the form $tp^e (t \geqq 1)$. Thus (1) is impossible.

Let us now consider

$$(3) \qquad \frac{(2n+1)!}{n!(n+1)!} = \binom{2n+1}{n} = 2\binom{2n+1}{k} = 2\,\frac{(2n+1)!}{k!(2n-k+1)!}$$

with $n>2$ and $0<k<n$. From (3) follows

$$(4) \qquad \frac{2n-k+1}{k+1}\cdot\frac{2n-k}{k+2}\cdot\cdots\cdot\frac{n+3}{n-1}\cdot\frac{n+2}{n} = 2.$$

Equation (4) is impossible for $k+1=n>2$, hence $n-k>1$. Further we can assume that $2n-k+1<2(k+1)$ and therefore $k+1>2(n-k)>2$. Applying again the theorem of Sylvester we now infer that in the set $k+1, \cdots, 2n-k+1$ there is a number having a prime divisor $q \geqq 2(n-k)+1>3$. A factor $q^g (g \geqq 1)$ occurs only once in the set, for we have $k+1+q \geqq 2n-k+2$. Multiplying (4) through by $n+1$ we get a contradiction in the same way as above provided that q does not divide $n+1$. If q divides $n+1$ then we have from (4)

$$2 = \prod_{i=k+1}^{n} \frac{2(n+1)-i}{i} \equiv \pm 1 \pmod{q},$$

which is impossible because $q>3$. This completes the proof.

Editorial Note. The Proposers suggest the more general problem of showing: for fixed rational a the equation

$$\binom{2n}{n} = a\binom{2n}{k}$$

has only a finite number of solutions.

Simultaneous Equations

4567 [1953, 716]. *Proposed by R. S. Underwood, Texas Technological College*

Prove that the following simultaneous equations in the four unknowns x, y, z, u have one and only one solution in terms of the parameters a, b, where $0<a<5$, $0<b<5$:

$$\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{5-a} + \frac{u^2}{5-b} = 1$$

$$(x - z - 4)^2 + (y - u - 2)^2 = 5.$$

Solution by Chih-yi Wang, University of Minnesota. Let the left member of the first given equation be denoted by $F(a, b)$. Since the right member is independent of a and b, by setting $\partial F/\partial a = 0$ and $\partial F/\partial b = 0$ we obtain, respectively

$$x/z = \pm a/(5 - a), \qquad y/u = \pm b/(5 - b).$$

Then the four unknowns must be of the following forms:

$$(1) \qquad x = ka, \qquad z = \pm k(5 - a), \qquad y = mb, \qquad u = \pm m(5 - b)$$

where k and m are the respective constants of proportionality whose values have to be determined. It follows from the second given equation, if the solution is real valued, that $x-z$ and $y-u$ must be independent of a and b respectively; we must, therefore, choose the negative signs of (1). By substituting (1), with negative signs for z and u, into the two given simultaneous equations, we get, after simplification,

$$(2) \qquad 5k^2 + 5m^2 = 1, \qquad 5k^2 - 8k + 5m^2 - 4m = -3.$$

By solving (2) we obtain two double roots, namely, $k = 2/5$ and $m = 1/5$. Hence the unique solution is $x = 2a/5$, $y = b/5$, $z = -2(5-a)/5$, $u = -(5-b)/5$.

Also solved by A. R. Hyde, D. C. B. Marsh, and the Proposer.

A De Moivre Quintic

4568 [1954, 51]. *Proposed by A. W. Walker, University of Toronto*

It can be shown (*e.g.* by taking y as independent variable) that if $y^5 - 5y^3 + 5y - 5x = 0$, then $(4-25x^2)y'' - 25xy' + y = 0$ (*Mathematical Gazette*, vol. 29, 1945, p. 223). Any three distinct roots of the above quintic must therefore be linearly related. Investigate this algebraically, and derive the following factored form of the quintic (readily verified by expansion):

$$(y - y_1)(y - y_2)(y - Ay_1 - Ay_2)(y + y_1 + Ay_2)(y + Ay_1 + y_2) = 0,$$

where $A^2 - A - 1 = 0$.

Solution by Louis Weisner, Hunter College, New York City. Substituting $y = z + z^{-1}$, we obtain $z^{10} - 5xz^5 + 1 = 0$. If we put

$$R_1 = \sqrt[5]{\frac{5x + \sqrt{25x^2 - 4}}{2}}, \qquad R_2 = \sqrt[5]{\frac{5x - \sqrt{25x^2 - 4}}{2}},$$

the values of the radicals being chosen so that $R_1R_2 = 1$, it follows easily that the roots of the quintic are $y_1 = \epsilon R_1 + \epsilon^4 R_2$, $y_2 = \epsilon^4 R_1 + \epsilon R_2$, $y_3 = \epsilon^2 R_1 + \epsilon^3 R_2$, $y_4 = \epsilon^3 R_1 + \epsilon^2 R_2$, $y_5 = R_1 + R_2$, where ϵ is a primitive fifth root of unity. Hence

$$y_3 = -y_1 - Ay_2, \qquad y_4 = -Ay_1 - y_2, \qquad y_5 = A(y_1 + y_2),$$

where $A = -\epsilon^2 - \epsilon^3$. It is readily verified that $A^2 - A - 1 = 0$. The five roots hav-

ing now been expressed in terms of y_1 and y_2, the factored form of the quintic is evident.

Also solved by R. H. Bruck, M. S. Klamkin, D. C. B. Marsh, Chih-yi Wang, Roscoe Woods, and the Proposer.

Editorial Note. The Proposer, after having submitted an algebraic solution, finds that $y = 2\cos\theta$, $x = \frac{1}{8}\cos 5\theta$ reduces the quintic to a trigonometric identity and provides an easy way of verifying the differential equation and the relations among the roots.

A Function Intermediate Between x^n and e^x

4569 [1954, 51]. *Proposed by D. J. Newman, Republic Aviation Corporation, Farmingdale, N. Y.*

Find explicitly a function $f(x)$ such that $f\{f(x)\}$ is of the order of magnitude of e^x. (In other words, find a function intermediate between x^n and e^x.)

I. *Solution by Edgar Reich, the Rand Corporation, Santa Monica, California.* Take $f(x)$ as $[e^x]$ when x is not an integer, and as $x+1/x$ when x is an integer. The brackets indicate the greatest integer function.

II. *Solution by I. N. Baker, Adelaide University, South Australia.* We can find a continuous solution and one, in fact, such that $f\{f(x)\} = e^x$, this being a particular case of the problem of the "iteration" of e^x. See U. T. Boedewadt, *Mathematische Zeitschrift*, vol. 49, p. 497. The method is due to Abel.

$f(x)$ may be written in the form

$$f(x) = g^{-1}\{g(x) + \tfrac{1}{2}\}$$

where $g(x)$ has the property $g(e^x) = g(x) + 1$ and where $g^{-1}(x)$ is the function inverse to $g(x)$. Define inductively

$$\exp[n, x] = \exp[1, \exp[n-1, x]],$$
$$\exp[0, x] = x, \qquad \exp[1, x] = e^x,$$
$$\log[n, x] = \log\log[n-1, x]$$
$$\log[0, x] = x, \qquad \log[1, x] = \log x.$$

Then a solution to the problem is yielded by

$$\begin{aligned} g(x) &= e^x - 1, && \text{for } -\infty < x \leqq 0 \\ &= x, && 0 < x \leqq 1 \\ &= \log[n, x] + n, && \exp[n, 0] < x \leqq \exp[n, 1] = \exp[n+1, 0], \\ & && n = 1, 2, 3, \cdots. \end{aligned}$$

Finally

$$\begin{aligned} f(x) &= \log(e^x + \tfrac{1}{2}), && \text{for } -\infty < x \leqq -\log 2, \\ &= e^x - \tfrac{1}{2}, && -\log 2 < x \leqq 0, \end{aligned}$$

$$= \exp\,[n, \log\,[n, x] + \tfrac{1}{2}], \qquad \exp\,[n, 0] < x \leqq \exp\,[n, \tfrac{1}{2}],$$
$$= \exp\,[n+1, \log\,[n, x] - \tfrac{1}{2}], \qquad \exp\,[n, \tfrac{1}{2}] < x \leqq \exp\,[n, 1],$$
$$n = 0, 1, 2, \cdots.$$

It is a simple matter to verify directly that $f(x)$ is a continuous function and that it satisfies the requirements of the problem.

k-Chromatic Graphs

4570 [1954, 52]. *Proposed by G. A. Dirac, King's College, London, England*

(1) If $d \geqq k \geqq 2$, show that there exist regular connected k-chromatic graphs of degree d and of arbitrarily high order.

(2) If $k \geqq 4$, construct a k-chromatic graph which does not contain a complete k-graph as a subgraph, and in which the degree of every node except one is $k-1$.

(See also problem 4526 [1954, 352].)

Solution by the Proposer. (1) If $k \geqq 2$ there exist regular connected k-chromatic graphs of degree k* and of every order: Let $G_1, G_2, \cdots, G_{2n}$ be $2n$ mutually disjoint complete k-graphs,† where n is as large as we please, and let the nodes of G_m be $a_{m1}, a_{m2}, \cdots, a_{mk}$, $(m = 1, 2, \cdots, 2n)$. For $1 \leqq i \leqq n$ and $1 \leqq j \leqq k$, join $a_{2i,j}$ to $a_{2i-1,j}$ or to $a_{2i+1,j}$ according as j is odd or even. (Here $a_{2n+1,j}$ is to mean a_{1j}.) The graph so obtained is k-chromatic, connected and regular of degree k.

If $d \geqq k \geqq 2$, starting with a graph Γ which is k-chromatic, connected and regular of degree $d-1$, we can construct a graph which is k-chromatic, connected and regular of degree d: Let $\Gamma_1; \Gamma_2; \cdots; \Gamma_d$, whose nodes are $a_{11}, a_{12}, \cdots, a_{1m}; a_{21}, a_{22}, \cdots, a_{2m}; \cdots; a_{d1}, a_{d2}, \cdots, a_{dm}$, respectively, be d mutually disjoint graphs, isomorphic to Γ in such a way that $a_{1i}, a_{2i}, \cdots, a_{di}$ correspond for $1 \leqq i \leqq m$. Let $b_1, b_2, \cdots, b_m$ be m new nodes, b_j being joined to $a_{1j}, a_{2j}, \cdots, a_{dj}$ for $1 \leqq j \leqq m$. The graph so obtained is k-chromatic, connected and regular of degree d.

This completes the proof of (1).

(2) Take two mutually disjoint complete $(k-1)$-graphs G_1 and G_2 and join one node a_1 of G_1 to one node b_1 of G_2. Then construct a new node c and join c to every other node except a_1 and b_1. The resulting graph is k-chromatic, does not contain a complete k-graph, and every node except c is of degree $k-1$.

(Note the theorem of I. F. Stone: *The only k-chromatic graph for $k \geqq 4$ in which the degree of every node is $k-1$ is the complete k-graph.*)

Also solved by R. L. Brooks and W. T. Tutte.

* A linear *graph* is the figure obtained by joining certain points or *nodes* by line segments. If it requires k colors to color the nodes so that no two nodes of the same color are connected by a line, the graph is *k-chromatic*. The *degree* of a node is the number of segments ending there. A *regular* graph has all its nodes of the same degree. The *order* of a graph is the total number of nodes.

† A *complete* k-graph has k nodes, each connected to every other. Hence it is regular, of degree k-1, and is k-chromatic.

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to E. P. Vance, Oberlin College, Oberlin, Ohio.

Algebra for College Students, 2nd edition. By J. R. Britton and L. C. Snively. New York, Rinehart & Co., Inc., 1954. xiii+474+Tables and Index, $4.25.

College Algebra. By H. G. Apostle. New York, Henry Holt & Co., 1954. xii+394 +Tables and Index, $4.50.

College Algebra, 3rd edition. By P. K. Rees and F. W. Sparks. New York' McGraw-Hill Book Co., Inc., 1954. xiii+414+Tables and Index, $4.25.

Understanding College Algebra. By E. R. Smith, S. Selby and M. Kleiman. New York, Dryden Press, 1954. xvi+505+Appendix, Tables and Index, $3.50.

The four books under review afford some choice in treatment but are fairly standard in content. Just how standard the material is can be gained by comparing them with texts of thirty years ago. Two such texts were found to differ largely in giving much shorter treatments of the elementary material. Evidently, modern texts are fighting a losing battle in closing the gap between high school preparation and college work.

Britton and Snively has 189 pages before reaching quadratics. Their book does a thorough job of ironing out high school deficiencies. Evidently, not even simple arithmetic is to be taken for granted. Students offering a poor unit of high school algebra could use this book. The more advanced parts are not treated quite as completely as in the other books also under review, but all essential topics are adequately presented. There is a short chapter on approximate numbers. Graphs are used more freely than is usual. A small feature is several pages given to formulas from geometry. Large numbers of worked problems are carefully discussed. Both Horner's Method and Successive Approximations are treated, but the brief treatments of the cubic and quartic equations found in the other three books are omitted. Students using this book can gain a thorough drill in the mechanics of algebra as well as a feeling that algebra can be highly useful. However, it gives little to lead one to think of algebra as an ever-expanding subject.

Apostle's book devotes 142 pages to reaching quadratics. However, the book has a new approach. It devotes the first thirty-eight pages to an axiomatic treatment of the fundamental operations. The author felt that not all of the treatment should be taken at once. Enough remains to give the student a strong feeling that algebra is quite as logical as geometry. There is plenty of room for differing on how this should be done, but the author should be commended for treating college students as adults even though they may be deficient in elementary mathematics. Certainly, high school material deserves a new approach in college. Repetition alone is not apt to solve the student's problem. The author distinguishes between a ratio and a fraction similar to Euclid Book V. The other

volumes make no such distinction. Unless a more complete semantic discussion is given, one feels tempted to agree with them. Mathematical Induction is treated as usual. The proof using the Principle of Contradiction is also presented as different from Mathematical Induction rather than merely a second proof. Hobson's Real Variables Volume 1 seems to take the latter view. This book has a number of worked examples, some of which are rather briefly handled. In arrangement the pages are a little crowded but not unduly so.

The book by Rees and Sparks has 148 pages before quadratics. The logical treatment is brief, quite in contrast with the Apostle book. It has numerous worked examples set forth on large uncrowded pages. A student should find the book easy reading. The authors stress the flexibility of the book. All of these items are easy on the teacher. An abundance of satisfactory problems gives one a feeling of the usefulness of the subject. The question should be raised if these pleasant features do not also cause the student to get a feeling that algebra is only a bundle of unrelated topics complete in themselves. This criticism can always be raised about the usual college algebra text.

The book by Smith, Selby and Kleiman is more between Apostle's and Rees and Sparks' in its logical introduction. There is a distinct effort to make the rules plausible before stating them. Rules stated when counter examples are not available can hardly impress a student. It has about as much elementary material as Rees and Sparks. Features not in the other books are the Sigma notation, the idea of the sequence and some notion of limits. There is a fair chapter on series if the teacher can teach it before running out of time. The teacher will find the proof of the binomial theorem by mathematical induction incomplete. Appendix A on computing π is obsolete and should be corrected and brought up-to-date. Over fifty brief references to the history of mathematics should be welcomed by the student. The problems and general treatment are quite well done.

All four of these books are quite usable. They have pointed up all too plainly the gradual decline in high school algebra. They have tried to meet the problem. They are strong on drill and learning by doing. Their departure from tradition has been slight. In view of the present trend to add something modern to elementary mathematics, they are conservative.

J. W. HURST
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Fundamentals of College Mathematics. By J. C. Brixey and R. V. Andree. New York, Henry Holt and Co., 1954. 609 pages. $5.90.

This book is a freshman text of the so-called integrated type. Taken substantially in its entirety, it is suitable for a two-semester five-hour course, but the material is so arranged that a proper selection could be a suitable basis for a shorter course emphasizing trigonometry, or analytic geometry, or college algebra.

The twenty-five chapters of the book seem to fall more or less into four main

divisions. The first of these comprises the first seven chapters, in which the emphasis is on application of algebra to geometric ideas. Starting with some algebraic review, the student is led from the function concept and rectangular co-ordinates through equations of loci and curve tracing; from slope of a line and slope of a curve, to limits, continuity, derivative, and the calculus of polynomials insofar as it applies to relative maximum and minimum, with application to curve tracing and word problems, motion in a straight line, and related time-rate problems.

The next six chapters, eight to thirteen, involve standard topics usually given in a college algebra course, culminating in a substantial chapter on statistics. This group of chapters begins, after a short discussion of exponents and scientific notation, with logarithms and logarithmic computation, and, after a glance at the topic of variation, proceeds through progressions, permutations, combinations, the binomial theorem and probability, to the chapter on statistics. Here emphasis is placed on mean, standard deviation, and the binomial distribution, with some discussion of quality control. Later in the book there is a chapter on use of log paper and method of least squares applied to fitting a straight line to a set of data. This chapter could well be studied in conjunction with the statistics chapter. Other algebraic topics as determinants and their use, and the algebra of complex numbers, are deferred to later chapters after the trigonometry and more geometry of straight lines have been developed.

The next five chapters of the book include standard topics in trigonometry emphasizing the analytic side and leading up to the chapters on analytic geometry in polar co-ordinates and complex numbers with the polar or vector point of view prominent.

There is a return to the calculus in a chapter involving some integration of polynomials with applications to problems of motion and areas. No introduction of the concept of differential is made, but the symbol of integration is used with the usual notation, $\int f(x)dx$. This presents the student with an unnecessary conceptual mystery, and the presentation suffers somewhat, as a result.

The remainder of the book, except for the algebraic portions referred to above, is devoted to conventional topics in analytic geometry which were not touched on in the first seven chapters: the geometry of the straight line, conics, parametric equations, and a chapter on the analytic geometry of space. There is a concluding chapter which is intended to stimulate student interest in mathematics. Here the selection of topics is wide and includes short discussions of matrix multiplication, angle trisection, magic squares, perfect number, Fermat's Last Theorem, the four-color problem, cryptography, base of a number system, and computing machines.

There are many commendable things that may be said for this volume. It is well written, from the point of view of both clarity and freshness; the problems are interesting, the format is good, and the diagrams well done. The book is inviting to read, and should be a stimulating start in the student's study of collegiate mathematics. The book is also a step in the right direction in satisfying

the need for early introduction of the calculus to engineering and scientific students so that a more substantial course in this basic science can be given in the sophomore year.

A unique feature of the book is found in the extensive skeleton solutions, with numerous sketches and graphs included in the answers to the odd problems. There is also a self test provided at the end of each chapter.

Here and there one might voice a slight note of objection. On page 13, the authors speak of "finding rational factors." Previously no mention has been made of the word *rational;* true, there is a footnote on the page, in which the definition of a rational number is implied, but this definition might well be introduced before using the term. In connection with the introduction of the imaginary numbers, the term *complex* is used as if referring exclusively to non-reals, and this is also implied in the analysis given on page 16. The idea of a discontinuity in a locus is discussed before any mention is made of continuity. One might also question the value of the exposure of freshmen to the ϵ, δ definition of a limit and might wonder at the failure to use previously-developed concepts of the calculus in the chapters on conics—for example, in showing the reflector property of the parabola. However, these are mere minor notes of discord in a volume which is, in this writer's opinion, a piece of work well done and a very welcome addition to the literature of collegiate mathematics.

NATHAN SCHWID
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Emerging Practices in Mathematics Education, Twenty-Second Yearbook of the National Council of Teachers of Mathematics, Washington, D. C., National Council of Teachers of Mathematics, 1954.

A valuable contribution to the literature on the teaching of mathematics, this book of articles was compiled under the editorship of John R. Clark with the cooperation of John Kinsella, Joy E. Mahachek, Phillip Peak, and Veryl Schult, and with the assistance of three members of the yearbook Committee of the National Council, *viz.*, Frank Lankford, W. D. Snader, and F. L. Wren. The book is composed of six parts, with the following respective self-explanatory titles: Various Provisions for Differentiated Mathematics Curriculums; Laboratory Teaching in Mathematics; Teacher Education; New Emphases in Subject Matter; The Evaluation of Mathematics Learning; and (annotated) Bibliography of "What Is Going on in Your School?"—1952–1953. Fifty-six different authors contributed, by invitation, the 51 articles distributed among the first five parts. The sixth part is a condensation of a department of "The Mathematics Teacher" over the period indicated in the title.

In Part One, 15 articles are arranged under five headings. Contributors discuss their analyses and/or successful experiences with problems associated with (1) reorganization of content, (2) homogeneous grouping of pupils, (3) effective classroom procedures where homogeneous grouping is undesirable or not feasible, (4) preparation of the public to accept and even contribute to changes

in a field usually regarded as static and inflexible, and (5) appropriate guidance procedures to accompany new programs. Part Two contains ten articles describing the use and merits of commercial and home- and pupil-made visual and other material aids to instruction, and of television and radio as media of instruction at various levels and for divers purposes. Part Three comprises 12 articles on a few of the many problems arising in connection with pre-service and in-service training of teachers. The discussions provide evidences of new awakenings in the areas of the induction of neophytes and of continuing professional growth among veteran teachers which must prove heartening to many who have long awaited these developments. In Part Four, new departures in teaching geometry, new mathematics, and approximate numbers are discussed. Part Five consists of seven articles; the evaluation procedures described convince one that some teachers are no longer content to rely solely on standardized tests nor even on their own time-tested home-made examinations. The manifestly deepening interest in the pupil as a dignified human being is encouraging. The list of Selected References following Part Six is a significant part of the book—indeed, some of the items whose titles it includes may prove more important than some of those in the body of the book.

The rarity of typographical and other errors compliments the industry of the authors and the diligence of the editors. However (page 321, line 12), 1.73 is a rather rough approximation to $\sqrt{2}$, and attention must be called, *e.g.*, to the first sentence of paragraph 3 on page 314 and to paragraph 2 on page 324.

In a book of this kind the several authors are expected to be in at least partial disagreement on some points. In their juxtaposition, the articles on approximate numbers in Part Four tend to fulfill and sharply emphasize this usual expectation.

In a period of almost explosive expansion in the number and diversity of marvelously interesting and worthwhile activities in which professional mathematicians are engaged, it is surprising and somewhat disappointing that the pertinent articles here devote as little attention as they do to prospective careers in mathematics. Perhaps this deficiency has been or can be corrected in the classroom or in the guidance procedures.

The criticisms offered are obviously constructive and certainly minor when compared with the high quality and good conception and execution of the book.

WADE ELLIS
Oberlin College

Analytic Geometry. By Gordon Fuller. Cambridge, Addison-Wesley Publishing Company, Inc., 1954. 205 pages. $3.85.

Analytic Geometry. By E. S. Smith, Meyer Salkover and H. K. Justice. New York, John Wiley and Sons, Inc., 1954. 306 pages. $4.00.

Both of these texts start in the usual way, giving the first two chapters to a discussion of rectangular coordinates and plotting, developing the elementary

formulae such as the distance between two points, $d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$ and the slope $m=(y_2-y_1)/(x_2-x_1)$. After developing these formulae which assume the straight line in Chapter 2, each takes up and develops the straight line equations in Chapter 3.

Smith, Salkover and Justice give a good discussion of the conic sections in Chapter 4 and, in the following chapters, treat the circle, parabola, ellipse and hyperbola as special conics. This is done in a logical way with considerable emphasis on the eccentricity. Transformation of coordinates is left to Chapter 9.

Fuller takes up transformation of coordinates in Chapter 4 and discusses the individual conics in succeeding chapters, placing more emphasis on the simplified forms. There is also a chapter on the slope of a curve which is a good introduction to the idea of the calculus.

Both discuss curve plotting from empirical data and both have a good selection of plane curves of degree higher than two as well as some curves in polar coordinates. Each gives a good discussion of space coordinates, brief, but sufficient for the purpose for which they were written.

There are lists of problems conveniently distributed through the texts and answers are given for the odd-numbered problems.

They are valuable additions to the list of publications in this subject.

C. O. WILLIAMSON
College of Wooster

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items should be submitted at least two months before publication can take place.

ANNUAL MEETING OF AMERICAN SOCIETY FOR ENGINEERING EDUCATION

The Annual Meeting of the American Society for Engineering Education will be held June 20–24, 1955, at Pennsylvania State University, State College, Pennsylvania. This meeting is attended each year by over 2000 leading educators and educational administrators from engineering colleges throughout the country. The Society this year wishes to extend a hearty welcome to guests of foreign nations who are interested in scientific and technological education. A program dealing with international aspects of engineering education will be included.

The Annual Meeting program this year will feature at one of its General Sessions the reports of two Society Committees. One of these—the Committee on Evaluation of Engineering Education—has been making a comprehensive

study of the objectives and curricular goals of engineering education, with particular emphasis on the changes which are needed to keep engineering education abreast of the advancing frontiers of science and technology. This Committee contains 40 members, who are leading engineering educators and educational administrators in colleges throughout the United States. The final report of this Committee will be presented at the 1955 Annual Meeting.

Another Committee report deals with the teaching of the humanities, the social sciences and the business studies in the engineering curriculum. The Committee is endeavoring to seek out those educational programs which show fresh and imaginative approaches which show promise of making exceptional contributions in terms of educational achievement. Since many engineers enter supervisory and managerial positions, this area of education is deemed of great importance in broadening the outlook and giving a cultural background to the engineer.

The Engineering College Research Council (ECRC) will have a General Session dealing with Education in Research. The Engineering College Administrative Council (ECAC) will feature a General Session dealing with Methods of Interesting Secondary School Students in Engineering as a Career. The ECAC and the ECRC will hold a combined dinner with a prominent speaker.

The ASEE conference will feature over 75 conferences dealing with all areas of engineering education, physics, mathematics, international relations, research, and others bearing upon the broad subject of engineering education.

All sessions will be open to foreign guests who wish to attend. The printed program for the Annual Meeting will be available about the end of April, 1955. For further information write to the Secretary of the ASEE, Northwestern University, Evanston, Illinois. For information regarding housing accommodations, write to Professor Kenneth Holderman, Pennsylvania State University, State College, Pennsylvania.

SUMMER SEMINAR OF THE CANADIAN MATHEMATICAL CONGRESS

The fifth Summer Seminar of the Canadian Mathematical Congress will be held at the University of Manitoba, Winnipeg, Manitoba, August 17 to September 9, 1955. The topic of this year's seminar is Analysis. Announcement will be made later of the titles of the Research Lectures. As in the past, instructional courses are designed for graduate students and as refresher courses for others; they assume little previous knowledge, but do assume some mathematical maturity. The object of the sub-seminar is to provide a forum for further treatment and discussion of the material of the Research Lectures.

The Research Lecturers, who will give two lectures a week, are: Heinrich Behnke, Westfälische Wilhelms Universität, Münster; W. K. Hayman, University College of the South West, Exeter, England; Jean Leray, Collège de France. The Sub-Seminar Leader is V. Hlavatý.

The following instructional courses will be given: Differential Equations by G. F. D. Duff, Queen's University; Combinatorial Analysis, N. Mendelsohn,

University of Manitoba; Theory of Transforms, D. B. Sumner, McMaster University.

It is expected that those travelling to the seminar by rail will travel at reduced fares within the Dominion of Canada. Lodging and board will be arranged at reasonable prices through the facilities of the University of Manitoba, and we hope to obtain additional accommodation for those desirous of attending with their families. Fee for the seminar is $10.

For further information please write to: Canadian Mathematical Congress, Chemistry Building, McGill University, Montreal, Canada.

SUMMER SCHOLARSHIP PROGRAM OF ELECTRODATA CORPORATION

ElectroData Corporation will continue the summer scholarship program which was inaugurated last year. The purpose of the program is to train graduate students of applied mathematics in industrial methods and techniques. Training in digital computing equipment and the organization and management of a corporation that produces this equipment will be provided for a period of ten weeks during the summer of 1955.

Students are selected upon requests of the departmental chairman. The scholarship is in the form of free grants to the universities involved.

For further information write to: Dr. Paul Brock, ElectroData Corporation, 717 North Lake Avenue, Pasadena 6, California.

RESEARCH CONFERENCE ON NUMBER THEORY

A Research Conference on Number Theory sponsored by the National Science Foundation will be held on June 22, 23 and 24, 1955, at the California Institute of Technology. The provisional arrangements include sessions on the following topics: 1. Applications of modular functions to number theory. 2. Problems of additive number theory. 3. The use of computing machines in number theory. 4. Diophantine equations. 5. Class field theory, quadratic forms, and the Riemann hypothesis in function fields.

Inquiries regarding the Conference should be addressed to Professor A. L. Whiteman, Chairman of the Organizing Committee, University of Southern California, Los Angeles 7, California.

PERSONAL ITEMS

Dr. Paul Brock, manager of ElectroData Corporation's technical services department, has been re-elected Chairman of the Southern California Section of the Society for Industrial and Applied Mathematics.

Associate Professor Edith R. Schneckenburger of the University of Buffalo represented the Association at the inauguration of Chancellor C. C. Furnas of the University of Buffalo.

Professor John von Neumann of the Institute for Advanced Study has been appointed by President Eisenhower to a five year term as a member of the

Atomic Energy Commission. The appointment must be confirmed by the Senate.

Alabama Polytechnic Institute announces the following: Mr. C. V. Aucoin, formerly a research fellow at the Institute, and Mrs. Clair Aucoin, previously a teaching fellow at the Institute, have been appointed to instructorships; Professor H. C. Wang is on leave of absence and is at the Institute for Advanced Study for the year.

Bucknell University reports: Assistant Professor W. K. Smith has been promoted to an associate professorship; Mr. Gregory Wulczyn has been appointed to an instructorship.

At Denison University: Associate Professor Marion Wetzel has been appointed Chairman of the Department of Mathematics; Instructor Andrew Sterrett, Jr., is directing the core course in general education in mathematics.

Ohio University reports the following: Mr. Howard Becksfort of Syracuse University has been appointed to an assistant professorship; Mr. F. C. DeSua of the University of Pittsburgh and Miss Nancy M. Scribano, formerly a student at Northwestern University, have been appointed to instructorships.

University of Arkansas announces that Miss Theresa Renner and Mr. L. R. Tappan have been appointed to instructorships.

At the University of British Columbia: Dr. R. R. Christian of Yale University has been appointed to an instructorship; Associate Professor Douglas Derry has been promoted to a professorship; Assistant Professors B. N. Moyls and W. H. Simons have been promoted to associate professorships; Professor Douglas Derry, who was on leave of absence for a year, has returned from France where he held a Canadian Government Overseas Fellowship; Professor S. A. Jennings is serving as Acting Head of the Department of Mathematics for the year 1954–55.

University of Kentucky reports the following: Dr. W. M. Faucett, formerly a teaching fellow at Tulane University, and Dr. J. D. Riley, previously mathematician at the Naval Ordnance Laboratory, Silver Spring, Maryland, have been appointed to assistant professorships; Mr. W. C. Swift has received a General Electric Fellowship and is working on Summability Theory under the direction of Professor V. F. Cowling.

Professor O. W. Albert of the University of Redlands has retired with the title Emeritus Professor.

Associate Professor B. H. Arnold has been granted a sabbatical leave from Oregon State College for the Spring term of 1955 and is at the University College of Wales, Aberystwyth, Wales.

Dr. Dean C. Benson of Iowa State College has been appointed to an assistant professorship at South Dakota School of Mines and Technology.

Dr. R. L. Blair of the University of California, Davis, has been promoted to an assistant professorship.

Mr. C. E. Buie, formerly a teacher at Pasadena City College, has been appointed to a professorship at Mt. San Antonio College.

Miss Mary J. Burns, previously a student at Duquesne University, has a position as a technician with Westinghouse Atomic Power Division, Bettis Field, Pittsburgh, Pennsylvania.

Assistant Professor L. J. Burton of Lake Forest College has been promoted to an associate professorship.

Mr. C. H. Cook, aerophysics engineer for Consolidated Vultee Aircraft, Fort Worth, Texas, has accepted a position as Senior Mathematician for the Glenn L. Martin Aircraft Corporation, Baltimore, Maryland.

Assistant Professor Trevor Evans of Emory University has returned after a two-year leave of absence at the Institute for Advanced Study and the University of Chicago.

Miss Catherine S. Feeley, formerly an auditor for Northern States Company, Inc., Chicago, Illinois, is now employed as a mathematician by Chicago Midway Laboratories, University of Chicago.

Mr. M. J. Forray of the Polytechnic Institute of Brooklyn is on leave of absence for the year 1954–55 and has a position as Senior Stress Analyst at Republic Aviation Corporation, Farmingdale, New York.

Miss Joyce B. Friedman of the Defense Department, Washington, D. C., has accepted a position as a mathematician with the ACF Electronics, Alexandria, Virginia.

Mrs. Ruth M. Frisch, formerly a graduate assistant at Syracuse University, is a teaching assistant at the University of Southern California.

Dr. T. M. Gallie, Jr., previously an assistant at Rice Institute, has been appointed to an instructorship at Duke University.

Mr. R. M. Gordon of the University of California has accepted a position as a mathematical analyst with the Lockheed Aircraft Corporation, Burbank, California.

Mr. E. P. Graney, formerly a research associate at the Willow Run Research Center, University of Michigan, has a position as a member of the technical staff of the Ramo-Wooldridge Corporation, Los Angeles, California.

Associate Professor A. E. Halteman of the University of Idaho is on leave of absence and is at the University of Oregon.

Assistant Professor H. M. Hughes of the University of California has accepted an appointment as an analytical statistician at the School of Aviation Medicine, Randolph Field, Texas.

Associate Professor D. H. Hyers of the University of Southern California has been promoted to a professorship.

Dr. Jack Indritz of the University of Minnesota has been appointed to an assistant professorship at Washington University.

Associate Professor W. J. Jaffe of the Newark College of Engineering has been promoted to a professorship.

Mrs. Carol R. Karp, previously of New Mexico College of Agriculture and Mechanic Arts, is now a graduate student at the University of California.

Mrs. Katharine B. Keppler, recently at Milwaukee-Downer Seminary, is

Head of the Department of Mathematics of the Masters School, Dobbs Ferry, New York.

Mr. R. B. Kiltie, formerly a student at Columbia University, is now a graduate student at New York University.

Mr. E. H. Kingsley, previously a mathematician at the Aerial Measurements Laboratory, Northwestern University, Technical Institute, is now an operations analyst for the United States Air Force, Eglin Air Force Base, Florida.

Assistant Professor L. C. Knight, Jr., of Muskingum College has been promoted to an associate professorship.

Mr. E. R. Lancaster, recently a mathematician with the I.B.M. Corporation, Poughkeepsie, New York, has been appointed to an instructorship at Newark College of Engineering.

Associate Professor J. A. Larrivee of the University of Vermont has been appointed to an associate professorship at Worcester Polytechnic Institute.

Assistant Professor Octave Levenspiel of the Department of Chemical Engineering of Oregon State College has been appointed to an assistant professorship in Chemical Engineering at Bucknell University.

Mr. D. B. MacMillan of Columbia University has accepted a position as Mathematician with the Knolls Atomic Power Laboratory, Schenectady, New York.

Assistant Profesor Dale Maness of Baylor University has been appointed to an instructorship at the University of Kansas.

Mr. P. J. McCarthy, previously an assistant at the University of Notre Dame, has been appointed to an instructorship at the University.

Professor C. J. McCormick of Illinois State Normal University has been appointed Acting Head of the Department of Mathematics.

Mr. D. T. Mitchell, previously a student at Wabash College, is now a graduate student and teaching assistant at Purdue University.

Dr. O. B. Moan, a quality control staff engineer at Hughes Aircraft Company, Culver City, California, has accepted a position as a research specialist with the Lockheed Aircraft Corporation, Van Nuys, California.

Mr. S. I. Neuwirth, formerly a biometrician and assistant to the Secretary of the Committee on Research, American Medical Association, Chicago, Illinois, has accepted a position as Director of the Statistics Division of Mutual Insurance Advisory Association, New York City.

Dr. E. E. Osborne of the University of Connecticut has accepted a position with the National Bureau of Standards, Los Angeles, California.

Mr. W. E. Pace of the Virginia Polytechnic Institute has been promoted to an assistant professorship.

Mr. P. B. Patterson of the University of Florida has been promoted to an assistant professorship.

Associate Professor C. R. Perisho of Nebraska Wesleyan University has been appointed to an instructorship at State Teachers College, Mankato, Minnesota.

Mr. H. R. Rouse, formerly a teaching fellow at Vanderbilt University, has been appointed to an instructorship at Elizabethtown College.

Miss Elsie C. Rump, previously a teacher at Madison High School, Madison, Kansas, is teaching at Curtis School, Wichita City Schools, Kansas.

Dr. J. U. Russell, recently a graduate assistant at the University of Illinois, has been appointed to an assistant professorship at Southwestern at Memphis.

Mr. W. H. Sawyer, formerly a student at the University of Oklahoma, is teaching at Chemawa Junior High School, Riverside, California.

Associate Professor E. B. Shanks of Vanderbilt University has been promoted to a professorship.

Mr. J. L. Spenceley, previously a teacher at Grand Haven High School, has been appointed to an instructorship at Flint Junior College, Michigan.

Dr. J. K. Sterrett, formerly a mathematician with the Naval Research Laboratory, Washington, D. C., has a position as Chief of Aeroballistics, Directorate of Test Operations, Air Force Armament Center, Eglin Air Force Base, Florida.

Mr. L. R. Stidham, previously an assistant at New Mexico College of Agriculture and Mechanic Arts, has accepted a position as a data analysis supervisor at the Telecomputing Corporation, White Sands Proving Ground, New Mexico.

Assistant Professor R. G. Stoneham of San Diego State College has accepted a position as Mathematician at the Radiation Laboratory of the University of California.

Associate Professor G. L. Tiller of Southwestern at Memphis has been appointed to an associate professorship at the University of Georgia, Atlanta Division.

Mr. J. W. Toole has been appointed to an instructorship at St. Peter's College, Jersey City, New Jersey.

Assistant Professor R. Z. Vause, Jr., of Clemson Agricultural College has been appointed to an assistant professorship at Memphis State College.

Assistant Professor Robert Weinstock of Stanford University has been appointed to an assistant professorship at the University of Notre Dame.

Mr. E. H. Weiss, formerly an engineer with the Engineering and Research Corporation, Riverdale, Maryland, has accepted a position as Mathematician with the M. W. Kellogg Company, Jersey City, New Jersey.

Mr. E. F. Whittlesey of Bates College has been appointed to an instructorship at Trinity College, Connecticut.

Dr. H. H. Wicke of Lehigh University has a position as a staff member of the Sandia Corporation, Albuquerque, New Mexico.

Assistant Professor R. E. Wild of the University of Idaho is a teaching assistant at the University of California, Los Angeles.

Mr. G. P. Williams, previously a statistician for the Army Ordnance, Cincinnati Ordnance District, Cincinnati, Ohio, is now an operations research analyst for General Electric Company, Cincinnati, Ohio.

Dr. R. J. Wisner of the University of British Columbia has been appointed to an assistant professorship at Haverford College.

Mr. W. B. Woolf, formerly a graduate student at Claremont Graduate School, is at the University of Michigan on a fellowship.

Assistant Professor J. W. Young of the University of Florida has accepted a position as a mathematician in the Applied Science Division, I.B.M. Corporation, Atlanta, Georgia.

Dr. R. E. Zink of Purdue University is now in military service.

Dr. G. A. Campbell, who had retired from his position with American Telegraph and Telephone Company, New York City, died on November 10, 1954. He was a charter member of the Association.

Emeritus Professor J. A. Caparo of the University of Notre Dame died on July 12, 1954. He was a charter member of the Association.

Miss Georgie T. Davis of Virginia Polytechnic Institute died on May 25, 1954.

Reverend J. A. Hearn of St. Joseph's College, Philadelphia, Pennsylvania, died on December 11, 1953.

Mrs. Kathryn B. Rolfe of Modesto, California, died on August 3, 1954.

Sister Mary Paula of Marygrove College died on October 7, 1954. She had been a member of the Association for twenty-one years.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

THE THIRTY-EIGHTH ANNUAL MEETING OF THE ASSOCIATION

The thirty-eighth annual meeting of the Mathematical Association of America was held at the University of Pittsburgh, Pittsburgh, Pennsylvania, on Thursday, December 30, 1954, in conjunction with the annual meetings of the American Mathematical Society, the Association for Symbolic Logic, and the Society for Industrial and Applied Mathematics. About seven hundred and twenty-five persons were registered, including the following four hundred and twelve members of the Association:

S. S. Abhyankar, J. E. Adney, Jr., M. I. Aissen, E. F. Allen, C. B. Allendoerfer, A. G. Anderson, R. D. Anderson, Mary L. C. Andrews, Nachman Aronszajn, R. A. Askey, W. F. Atchison, Miriam C. Ayer, J. L. Bagg, W. L. Baily, Jr., Joshua Barlaz, Wallace E. Barnes, J. H. Barrett, J. B. Bartoo, E. H. Batho, J. D. Baum, W. R. Baum, Thomas Bauserman, Helen P. Beard, R. A. Beaumont, E. G. Begle, Richard Bellman, C. P. Benner, A. A. Bennett, A. P. Berens, R. R. Bernard, Dorothy L. Bernstein, R. J. Bickel, R. H. Bing, Garrett Birkhoff, B. H. Bissinger, Warren Blaisdell, I. E. Block, Joseph Blum, J. O. Blumberg, R. O. Blummer, Jr., R. D. Boswell, Jr., S. G. Bourne, J. W. Brace, W. G. Brady, Leila Dragonette Bram, J. L. Brenner, Paul Brock, Foster Brooks, J. L. Brown, Jr., A. M. Bryson, R. C. Buck, R. S. Burington, L. E. Bush, A. T.

Butson, R. K. Butz, J. E. Byrne, E. J. Camp, Dorothy I. Carpenter, W. B. Carver, Maria Castellani, Jeremiah Certaine, J. O. Chellevold, Y. W. Chen, S. S. Chern, T. S. Chihara, J. G. Christiano, C. E. Clark, W. E. Cleland, K. J. Cohen, L. W. Cohen, Geraldine A. Coon, Byron Cosby, Jr., R. R. Coveyou, V. F. Cowling, H. S. M. Coxeter, E. H. Crisler, D. W. Crowe, A. B. Cunningham, J. C. Currie, H. B. Curry, J. H. Curtiss, Elizabeth H. Cuthill, Wayne Dancer, R. B. Davis, R. L. Davis, R. B. Deal, Jr., C. H. Denbow, W. E. Deskins, A. H. Diamond, S. F. Dice, Mary P. Dolciani, Jesse Douglas, T. C. Doyle, W. L. Duren, Jr., W. H. Durfee, P. S. Dwyer, J. C. Eaves, Albert Edrei, P. D. Edwards, R. D. Edwards, Samuel Eilenberg, F. M. Ellis, D. H. Erkiletian, Jr., M. H. M. Esser, H. J. Ettlinger, Trevor Evans, Ky Fan, F. A. Ficken, C. D. Firestone, W. T. Fishback, Gloria C. Ford, M. K. Fort, Jr., J. S. Frame, C. H. Frick, Otha Fuller, Jr., R. E. Fullerton, J. W. Gaddum, T. M. Gallie, Jr., G. N. Garrison, H. M. Gehman, B. H. Gere, K. G. Getman, W. M. Gilbert, Leonard Gillman, Wallace Givens, R. D. Glauz, H. W. Godderz, F. S. Goepper, Jr., Michael Goldberg, Samuel Goldberg, S. I. Goldberg, J. K. Goldhaber, A. W. Goodman, W. O. Gordon, S. H. Gould, Arthur Grad, W. W. Graham, E. L. Grindall, Emil Grosswald, Arnold Grudin, R. R. Gutzman, W. T. Guy, Jr., V. H. Haag, Franklin Haimo, D. W. Hall, Marshall Hall, Jr., H. W. Handsfield, Frank Harary, Gerald Harrison, O. G. Harrold, Jr., H. H. Hartzler, G. A. Hedlund, C. E. Heilman, R. G. Helsel, Melvin Henriksen, R. T. Herbst, Coleman Herpel, I. N. Herstein, Archie Higdon, T. H. Hildebrandt, J. J. L. Hinrichsen, I. I. Hirschman, Jr., J. G. Hocking, L. Aileen Hostinsky, D. B. Houghton, Chuan-Chih Hsiung, G. B. Huff, Ralph Hull, M. Gweneth Humphreys, C. A. Hutchinson, M. A. Hyman, Jane C. Ingersoll, R. Y. Iwanchuk, R. F. Jackson, S. B. Jackson, R. D. James, Evan Johnson, Jr., L. W. Johnson, R. E. Johnson, R. P. Johnson, F. B. Jones, John Jones, Jr., P. S. Jones, Harriett R. Junior, F. E. Justis, Irving Kaplansky, Chosaburo Kato, J. B. Kelly, L. M. Kelly, J. G. Kemeny, J. R. F. Kent, D. E. Kibbey, S. C. Kleene, George Klein, L. C. Knight, Jr., J. C. Knipp, F. T. Kocher, Jr., J. D. E. Konhauser, H. L. Krall, H. C. Kranzer, J. B. Kruskal, Jr., A. H. Kruse, Harold W. Kuhn, Stephen Kulik, R. G. Kuller, R. J. Lambert, E. H. Larguier, George Laush, W. S. Lawton, E. B. Leach, A. B. Lehman, Marguerite Lehr, H. R. Leifer, Walter Leighton, W. W. Leutert, Norman Levine, F. H. Lloyd, R. W. Long, Lee Lorch, D. B. Lowdenslager, L. L. Lowenstein, C. I. Lubin, G. R. MacLane, Saunders MacLane, H. M. MacNeille, C. G. Maple, R. W. Marsh, Gloria A. Martin, W. T. Martin, K. O. May, John McCarthy, Myles McConnon, Dorothy McCoy, S. W. McCuskey, R. G. McDermot, J. H. McKay, E. J. McShane, A. E. Meder, Jr., R. E. Messick, D. D. Miller, W. I. Miller, Josephine M. Mitchell, E. E. Moise, Deane Montgomery, R. A. Moore, T. W. Moore, C. B. Morrey, Jr., Max Morris, L. T. Moston, B. H. Mount, Jr., C. W. Munshower, W. L. Murdock, F. H. Murphy, J. R. Musselman, E. F. Myers, Zeev Nehari, Morris Newman, J. A. Nohel, E. S. Northam, E. P. Northrop, A. B. J. Novikoff, C. O. Oakley, E. N. Oberg, Ruth E. O'Donnell, M. M. Ohmer, M. W. Oliphant, H. W. Oliver, Emma J. Olson, F. R. Olson, L. A. Ondis, II, J. H. Oppenheim, Morris Ostrofsky, J. C. Oxtoby, J. A. Painter, F. P. Palermo, T. P. Palmer, T. K. Pan, O. O. Pardee, B. C. Patterson, G. W. Patterson, III, P. A. Penzo, F. W. Perkins, W. J. Pervin, I. D. Peters, Mary Pettus, H. R. Phalen, T. J. Pignani, C. F. Pinzka, Everett Pitcher, R. J. Pitts, J. C. Polley, J. W. Popow, J. T. Powers, G. B. Price, R. E. Priest, Valdemars Punga, A. L. Putnam, Tibor Rado, J. F. Randolph, L. T. Ratner, G. E. Raynor, Edgar Reich, Irving Reiner, Haim Reingold, Eric Reissner, A. C. Reynolds, Jr., D. E. Richmond, J. D. Riley, R. F. Rinehart, L. A. Ringenberg, T. J. Rivlin, J. H. Roberts, Fred Robertson, G. deB. Robinson, Louis Robinson, L. V. Robinson, R. A. Rosenbaum, Alex Rosenberg, P. C. Rosenbloom, Louis Ross, J. B. Rosser, J. P. Roth, W. C. Royster, Mary E. Rudin, H. J. Ryser, Louis Sacks, E. A. Saibel, Charles Saltzer, A. C. Schaeffer, R. D. Schafer, Robert Schatten, Edith R. Schneckenburger, K. C. Schraut, B. L. Schwartz, W. R. Scott, C. F. Sebesta, George Seifert, Samuel Selby, D. H. Shaffer, A. L. Shields, R. L. Shively, Marlow Sholander, Edward Silverman, Annette Sinclair, Sister Mary Deborah, Sister Marie Gertrude, M. F. Smiley, E. C. Smith, Jr., W. S. Snyder, E. V. Somers, T. H. Southard, G. L. Spencer, II, Vivian E. Spencer, R. H. Spohn, C. E. Springer, F. H. Steen, H. E. Stelson, Andrew Sterrett, Jr., J. K. Sterrett, Guy Stevenson, F. M. Stewart, R. R. Stoll, M. W. Stone, Alexander Strasser, D. D. Strebe, Nikola Svilokos, J. D. Swift, W. R. Talbot,

T. T. Tanimoto, J. S. Taylor, Mildred E. Taylor, William Clare Taylor, Jean E. Teats, G. B. Thomas, Jr., J. M. Thomas, L. O. Thompson, D. L. Thomsen, Jr., R. M. Thrall, John Todd, M. L. Tomber, Leonard Tornheim, W. R. Transue, Halsey K. Truan, A. W. Tucker, E. P. Vance, R. S. Varga, Mary C. Varnhorn, Bernard Vinograde, S. I. Vrooman, T. L. Wade, Jr., G. L. Walker, Marcelle M. Walker, J. A. Ward, W. H. Warner, G. C. Webber, J. V. Wehausen, Brother Bernard Alfred Welch, C. P. Wells, D. W. Western, F. J. Weyl, C. R. White, P. M. Whitman, G. T. Whyburn, Albert Wilansky, W. L. Williams, R. A. Willoughby, D. M. Young, Jr., J. W. T. Youngs, A. D. Ziebur, J. A. Zilber, R. E. Zindler, R. E. Zink.

Sessions of the Association were held on Thursday morning and afternoon in the Auditorium of the Mellon Institute of the University of Pittsburgh. Vice-President H. S. M. Coxeter presided at the morning session and President E. J. McShane at the afternoon session. The Program Committee for the meeting consisted of Irving Kaplansky, Chairman; H. L. Meyer, Jr., and P. C. Rosenbloom.

FIRST SESSION OF THE ASSOCIATION

Session on Foundations of Geometry

"Geometries and incidence matrices," by Professor H. J. Ryser, Ohio State University.

"Finite projective planes," by Professor Marshall Hall, Jr., Ohio State University.

"The geometries of Hjelmslev," by Professor Reinhold Baer, University of Illinois.

SECOND SESSION OF THE ASSOCIATION

"Classroom notes," by Professor M. F. Smiley, State University of Iowa.

"Mathematics on television (demonstration and discussion)," by Professor P. S. Jones, University of Michigan. Discussants: Professors F. G. Fender, Rutgers University, Marguerite Lehr, Bryn Mawr College, and Fred Robertson, Iowa State College.

MEETING OF THE BOARD OF GOVERNORS

The Board of Governors of the Association met on Wednesday morning in the Lounge of the Hotel Webster Hall in Pittsburgh, with twenty-one members present. Among the more important items of business transacted were the following:

Professor G. B. Price of the University of Kansas was elected Second Vice-President for 1955–1956.

Professor D. E. Richmond of Williams College was elected Representative on the Policy Committee for Mathematics for 1955–1957.

Approval was given by the Board to the appointment by President McShane of the following Nominating Committee for 1955: A. W. Tucker, Chairman; Ralph Hull, and D. E. Richmond.

It was voted to hold the Thirty-seventh Summer Meeting at the University

of Washington, Seattle, Washington, on August 20–21, 1956, and the Fortieth Annual Meeting at the University of Rochester, Rochester, New York, on December 29, 1956. The Board also voted to approve the establishment of a joint committee with the American Mathematical Society to study a possible change in the time of holding the annual meetings of the two organizations.

In order to decrease the back-log of unpublished papers now in the hands of the Editor, the Board voted to authorize the printing of thirty-two additional pages of the MONTHLY during 1955.

ANNUAL BUSINESS MEETING OF THE ASSOCIATION

The annual business meeting of the Association was held on Thursday, December 30, 1954, at 2:00 p.m., in the Auditorium of the Mellon Institute of the University of Pittsburgh, Pittsburgh, Pennsylvania. President E. J. McShane presided.

The Secretary announced the results of the balloting for officers, in which 1299 votes were cast: W. L. Duren, Jr., of Tulane University was elected President for the two-year term 1955–1956. H. W. Brinkmann of Swarthmore College and M. A. Zorn of Indiana University were elected Governors for the three-year term 1955–1957.

MEETINGS OF OTHER ORGANIZATIONS

Sessions of the American Mathematical Society began on Monday, December 27 and continued through Wednesday afternoon. The Gibbs lecturer was Professor K. O. Friedrichs, and invited addresses were delivered by Professors Samuel Eilenberg and Lipman Bers.

The Association for Symbolic Logic met on Wednesday morning and afternoon, with an invited address by Professor G. H. von Wright.

A meeting of the Society for Industrial and Applied Mathematics was held on Tuesday evening.

ARRANGEMENTS, ENTERTAINMENT, AND RECREATION

The Committee on Arrangements for the meeting consisted of: J. C. Knipp, Chairman; J. O. Blumberg, A. M. Bryson, L. W. Cohen, H. M. Gehman, George Laush, Norman Levine, E. F. Myers, J. S. Taylor, Jean Teats.

Registration headquarters was in the lobby of Hotel Webster Hall. Persons attending the meeting were housed in that hotel and other Pittsburgh hotels. Lunch was served in the Faculty Club in the Cathedral of Learning and the Tuck Shop was available for light refreshments.

Tea was served in the Faculty Club on Monday afternoon by the ladies of the Mathematics Department. The Nationality Classrooms were open for inspection during the tea. A conducted tour of the Mellon Institute was held on Tuesday afternoon. A three-hour tour of the city of Pittsburgh took place on Wednesday morning.

A banquet for members of the mathematical organizations and their guests was held on Wednesday evening at Hotel Webster Hall. Professor J. S. Taylor was toastmaster. Speakers were Dean S. C. Crawford of the University of Pittsburgh; President H. W. Kuhn of the Society for Industrial and Applied Mathematics; President W. V. Quine of the Association for Symbolic Logic; President E. J. McShane of the Mathematical Association of America, and Associate Secretary L. W. Cohen of the American Mathematical Society.

At the banquet, Professor C. B. Allendoerfer presented a resolution of thanks to our host, the University of Pittsburgh, on behalf of the four mathematical organizations, which was adopted by a rising vote. Particular thanks are to be given to the local members of the Committee on Arrangements which did such an excellent job in organizing this interesting and valuable meeting.

H. M. GEHMAN, *Secretary-Treasurer*

THE OCTOBER MEETING OF THE OKLAHOMA SECTION

The annual meeting of the Oklahoma Section of the Mathematical Association of America was held on October 29, 1954, at Oklahoma City University, Oklahoma City, Oklahoma.

There were one hundred thirty-eight in attendance, including the following fifty members of the Association:

R. V. Andree, Arthur Bernhart, E. N. Brandt, Jr., J. C. Brixey, H. N. Carter, N. A. Court, R. B. Deal, Jr., R. C. Dragoo, I. E. Glover, B. T. Goldbeck, Jr., E. V. Greer, O. H. Hamilton, T. J. Head, E. E. Heimann, J. E. Hoffman, W. N. Huff, R. A. Hultquist, P. W. M. John, L. W. Johnson, J. T. Krattiger, J. E. LaFon, W. T. Lee, Gene Levy, H. W. Linscheid, Dora McFarland, B. L. Mackin, G. E. Meador, Dorothea Meagher, R. R. Murphy, Leonard Nichols, F. J. Palas, T. K. Pan, D. L. Patten, C. M. Pirrong, Jr., E. C. Rice, W. A. Rutledge, Herbert Scholz, Jr., J. W. Sehestedt, D. R. Shreve, M. G. Shults, H. W. Smith, O. S. Spears, C. E. Springer, Vivian Spurgeon, Margaret O. Taylor, J. D. Thomas, R. W. Veatch, G. R. Vick, Wesley Whittlesey, II, J. H. Zant.

At the business meeting, it was decided that the present arrangement of holding meetings of the Oklahoma Section of the Mathematical Association of America in conjunction with the annual meeting of the Mathematics Section of the Oklahoma Education Association and the National Council of Teachers of Mathematics did not provide enough time for presented papers. The members voted to continue holding a meeting in conjunction with the O.E.A., but to restrict papers at this meeting to those of reasonable interest to high school teachers, and to hold a second meeting on a different date at which research papers would be welcomed.

The following officers were elected for the year 1954–1955: Chairman, Professor R. R. Murphy, Panhandle Agricultural and Mechanical College; Vice-Chairman, Professor C. M. Pirrong, Oklahoma City University; Secretary-Treasurer, Professor R. V. Andree, University of Oklahoma.

The following papers were presented:

1. *Series related to the Fourier coefficients*, by Professor Simon Green, University of Tulsa.

If $f(t)$ is Lebesgue integrable $0 \leq t \leq 2\pi$, then the series $\sum a_n/n$ converges where a_n is the Fourier coefficient of $f(t)$ with respect to the function $\pi^{-1/2} \sin nt$. It is required, when n is a fixed number, that the coefficients ξ_i of the Fourier series are determined in such a way that $s_n(t)$ in the space L^2 approaches as close as possible $f(t)$, which leads to Bessel's inequality:

$$\sum_{i=1}^{n} c_i^2 \leq \int_a^b f^2(t)dt;$$

therefore $\sum c_i^2$ is convergent. It is shown that $\sum_{i=1}^{n} \sqrt{\bar{a}_i \bar{b}_i}$ is convergent if $\sum \bar{a}_i$ and $\sum \bar{b}_i$ are convergent. By replacing $\bar{a}_i$ by a_i^2 and $\bar{b}_i$ by $1/n^2$ it is concluded that $\sum_{n=1}^{n} a_n/n$ is convergent.

2. *On the combination of two bi-variate normal dispersion patterns relating to artillery fire*, by Major O. S. Spears, Combat Development Department, The Artillery School, Fort Sill, Oklahoma.

An artillery weapon is pointed at point O_1, a point in the target area which is the center of impact for a very large number of projectiles. The dispersion pattern of the weapon around this origin O_1 is a normal, bi-variate circular pattern; *i.e.*, the standard deviation of the fall of projectiles along any given line through O_1 is the same as that for any other line through the point. When each projectile explodes, the fragment spray is such that the probability-of-casualty function is also a bi-variate, normal distribution, with center at O_{2i}, the horizontal projection of the burst center for the i-th projectile. A method is developed for determining the expected fraction of casualties for n projectiles, and for a stipulated target area, when the standard deviations of the two previously mentioned bivariate probability density functions are given.

3. *The price of consistency*, by Professor J. B. Giever, University of Oklahoma, introduced by the Secretary.

The paper is a semi-expository discussion of the theorem that every field has an algebraically closed extension. A very simple, but essentially incorrect, proof is presented and several other proofs are outlined and considered from the point of view that the additional complications may be considered the price paid in order to work in a consistent mathematics. A proof is given which approximates the simplicity of the incorrect proof. Some ways of correcting the first proof are considered.

4. *Some interesting indecomposable continua*, by Professor O. H. Hamilton, Oklahoma Agricultural and Mechanical College.

The standard definition of an indecomposable continuum is presented and some of the properties of indecomposable continua derived by Kuratowski, Janizewski, and Moore are discussed. A special type of indecomposable continuum, the pseudo-arc as defined by Moise, is examined in detail and its property of being homogeneous, as discovered by Bing, is discussed.

5. *Concavity of curves relative to a point*, by Miss Ella L. Clement, Douglas High School, Oklahoma City, Oklahoma, introduced by the Secretary.

The intuitive notion of concavity, inward or outward, of an arc of a curve relative to a point is formulated mathematically in a manner analogous to the mathematical formulation of the notion of concavity, upward or downward, of an arc of a curve relative to a line. A theory of such concavity analogous to that employed in studying concavity of curves relative to a line is developed. Necessary and sufficient conditions for an arc of the graph of a function of a somewhat restricted class of functions $r=f(\theta)$ to be concave inward or outward in a θ-interval are established.

Necessary conditions are established as well as sufficient conditions. The theory is then employed to simplify the study of relative maxima and minima, points of inflection and curvature of polar loci. As an incident to the investigation, necessary and sufficient conditions that any of the polar coordinates of a point on the locus of $r=f(\theta)$ satisfy the equation are established.

6. *Similar triangles represented by points*, by Professor Arthur Bernhart, University of Oklahoma.

A triangle with three degrees of freedom may be represented by a point. Several such correspondences are considered. Triangles of a given shape generate simple point loci.

7. *A report on the 1954 Summer Conference in Collegiate Mathematics, Chapel Hill, North Carolina*, by Professor J. H. Zant, Oklahoma Agricultural and Mechanical College.

The conference heard Professor E. Artin lecture on Abstract Algebra and Professor T. Rado on Rigid Surfaces; Professors Tucker, Thrall, and Griffin conducted discussion groups. The aim to foster the improvement of undergraduate mathematics by discussing the meaning and development of modern mathematics implies that its basic concepts should be included in the undergraduate curriculum. Such a need is recognized by all in graduate school mathematics. The conference did not spell out specifically in even one instance or course how this should be done. This must be done either by future conferences or by professors in the colleges. Many will need help in doing this.

8. *What mathematics is wanted by the petroleum industry*, by Dr. D. R. Shreve, The Carter Oil Company, Research Laboratory, Tulsa, Oklahoma.

The mathematics used in economics, transportation, refining, exploration, and production problems in the petroleum industry was discussed. Emphasis was primarily on numerical mathematics of use in routine calculations and on the more modern numerical analysis required with high speed electronic computers. The need for mass production techniques, greater knowledge of numerical integration procedures, and understanding of computational stability was discussed.

R. V. ANDREE, *Secretary*

THE FALL MEETING OF THE MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA SECTION

The fall meeting of the Maryland-District of Columbia-Virginia Section of the Mathematical Association of America was held at Georgetown University, Washington, D. C. on December 4, 1954. Professor C. H. Frick, Chairman of the Section, presided at the morning and afternoon sessions.

There were one hundred thirty persons in attendance, including the following eighty-two members of the Association:

J. C. Abbott, D. F. Atkins, R. P. Bailey, M. P. Berri, Evelyn Boyd, J. W. Brace, J. L. Brenner, B. H. Buikstra, G. H. Butcher, W. E. Byrne, H. H. Campaigne, J. F. Canu, G. R. Clements, E. W. Coffin, L. W. Cohen, C. H. Cook, Elizabeth H. Cuthill, J. A. Duerksen, R. P. Eddy, P. J. Federico, E. J. Finan, Gloria C. Ford, C. H. Frick, Michael Goldberg, R. A. Good, E. S. Grable, E. C. Gras, D. W. Hall, J. E. Houle, Jr., M. Gweneth Humphreys, Louise S. Hunter, J. E. Ikenberry, Jane C. Ingersoll, S. B. Jackson, F. E. Johnston, L. M. Kells, C. F. Koehler, A. E. Landry, D. B. Lloyd, J. J. MacDonnell, G. J. Mann, Ella C. Marth, M. H. Martin, R. M. Mason, E. S. Mayer, Carol V. McCamman, E. J. McShane, Florence M. Mears, Joseph Milkman, L. I. Mishoe, A. K. Mitchell, R. W. Moller, T. W. Moore, W. H. Norris, E. P. Northrop, M. W. Oliphant, W. W.

Proctor, O. J. Ramler, R. W. Rector, J. N. Rice, Sara L. Ripy, W. G. Saunders, J. W. Sawyer, Veryl G. Schult, Paul Shapiro, W. F. Shenton, C. H. Sisam, W. S. Soar, G. L. Spencer, II, C. F. Stephens, W. J. Strange, Choy-Tak Taam, Feodor Theilheimer, J. A. Tierney, John Todd, Marian M. Torrey, John Tyler, C. H. Wheeler, III, P. M. Whitman, G. T. Whyburn, G. T. Williams, D. M. Young, Jr.

The Chairman announced the appointment of the High School Contest Prize Committee: Mr. P. L. Chessin, Mr. R. O. Blummer, and Professor C. H. Wheeler, III. The Secretary reported the addition of four names to the list of volunteer lecturers in the section's Undergraduate Lecture Program: Professor R. C. Yates, Virginia Polytechnic Institute; Professor W. K. Morrill, Johns Hopkins University; Dr. W. W. Leutert, Ballistics Research Laboratory, Aberdeen Proving Ground; and Dr. E. P. Northrop, National Science Foundation. The Section, by a unanimous vote, expressed its appreciation to the members of the High School Contest Committee (Chairman, Professor W. H. Norris) for their excellent work.

The following papers were presented:

1. (A) *Evaluation of a certain finite sum;* (B) *A case of superiority of numerical integration over integration in closed form,* by Professor C. L. Beckel, Georgetown University, introduced by Professor Oliphant.

(A) The sum $f(f-1)\cdots(f-m)\sum_{k=0}^{m}(-1)^k(m/k)(f-k)^{-1}$, arising in a problem in molecular structure, was considered. It was shown that the sum is $(-1)^m m!$, independent of f.

(B) A certain set of definite integrals was encountered in the same molecular problem. Performing the integrations led to a double sum of terms in which the signs alternated. In the process of summing all significant figures might be lost. However, a new formulation led to numerical integrations giving answers having the desired degree of precision.

2. *On the uniform convergence of a certain eigenfunction series,* by Professors L. I. Mishoe and Gloria C. Ford, Morgan State College, presented by Miss Ford.

The problem of expanding a function $F(x)$ of bounded variation on (0, 1) in terms of the eigenfunctions $u_n(x)$ of the equation $u''+q(x)u+(p(x)u-u')=0$, satisfying the boundary conditions $u(0)=u(1)=0$, was considered by Friedman and Mishoe. In the present paper it was proved that the series $\sum_{-\infty}^{\infty} a_n u_n$ converges uniformly to $F(x)$ in the open interval (0, 1) provided $F(0)=F(1)=0$, and $F(x)$ has a continuous first derivative on (0, 1). The condition is sufficient but not necessary.

3. *On certain solutions of* $B(y)u_{xx}+u_{yy}=0$, by Professor E. C. Watters, United States Naval Academy, introduced by the Secretary.

Consideration of certain problems in fluid dynamics led to the study of solutions to the partial differential equation

$$B(y)u_{xx} + u_{yy} = 0, \qquad B(y) \neq \text{constant},$$

defined implicitly by $U(u)=X(x)+Y(y)$. For solutions of this type to exist which are not obtainable by ordinary separation of variables, it is necessary that $B(y)$ satisfy a certain differential equation. Conversely, given $B(y)$ satisfying the latter differential equation, a method for obtaining the solutions of the partial differential equation is outlined. The solution of this problem depends partially on results obtained by M. H. Martin.

4. *Hitting the mark*, by Mr. G. T. Williams, Operations Research Office, Johns Hopkins University.

A marksman fires at a target, with a constant probability p of hitting it. It takes him time t' between rounds, to aim and fire. If this time is distributed according to a known function $f(t)$, and $F(t)$ gives the distribution of time it actually takes to hit the target, it was found that

$$F(t) = pf(t) + (1 - p)\int_0^t f(\tau)F(t - \tau)d\tau.$$

A simple calculation then showed that

$$pM_n = \mu_n + (1 - p)\sum_{k=1}^{n-1}\binom{n}{k}\mu_k M_{n-k},$$

where $M_n = \int_0^\infty t^n F(t)dt$ and $\mu_n = \int_0^\infty t^n f(t)dt$. It was pointed out that the case $n = 1$ is of particular interest, since the conclusion may be drawn that the mean time it takes to hit the target is equal to the mean number of rounds fired multiplied by the mean length of the rounds. A brief discussion was also given of the explicit solution of the cognate problem where $f(t)$ is the normal distribution.

5. *The discriminant of a polynomial of special type*, by Professor J. E. Houle, Georgetown University.

Given $f(x) = x^n + a_1x^{n-1} + \cdots + a_n$, with the a_i in an algebraic extension field $F(\theta)$ where the defining equation of θ is of degree m, Finan (this MONTHLY, vol. 45) discussed a matrix method of obtaining an equation $h(x) = 0$ over F among whose roots occur all the roots of $f(x) = 0$. Let C_f be the companion matrix of $f(x)$, and let B denote the compound matrix obtained from the matrix $A = f'(C_f)$ by a transformation similar to Finan's. It was shown that the discriminant of $h(x) = 0$ is $(-1)^{nm(nm+1)}|B|a^2/2$, where a is in $F(\theta)$. This involves extending certain results due to MacDuffee (this MONTHLY, vol. 57).

6. *Report on the High School Mathematics Contest*, by Mr. W. H. Norris, Maury High School, Norfolk, Virginia.

Mr. Morris, Chairman of the High School Contest Committee, reported the results of the first Contest, held in May 1954. Dr. D. B. Lloyd served as Chairman of the Test Committee, Dr. F. E. Johnston served as Chairman of the Board of Reviewers. About 1800 students in more than 40 high schools participated. The following results were announced: *First Place*, Thomas B. Ballard, Baltimore Polytechnic Institute; *Second Place*, Courtney W. Doyle, Baltimore Polytechnic Institute; *Regional Winners: Maryland*, Thomas B. Ballard, Baltimore Polytechnic Institute; *District of Columbia:* Frederick Hoffman, Calvin Coolidge High School, Washington, D. C.; *Virginia*, Paul E. Heath, Jr., Norview High School, Norfolk, Virginia.

7. *The circulation index in topological analysis*, by Professor G. T. Whyburn, University of Virginia.

Professor Whyburn discussed recent results and methods in the program of developing and proving by topological methods those theorems in analysis, particularly the ones concerned with functions of a complex variable, which are essentially topological in character. A procedure was outlined for obtaining, with the aid of the circulation index of a mapping about a point, the fundamental properties of lightness and openness for a non-constant analytic function as a special case of the theorem which asserts that these properties belong to any non-constant function which is the limit of a uniformly convergent sequence of differentiable functions.

R. P. BAILEY, *Secretary*

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PERIODS OF SERVICE OF FORMER OFFICERS OF THE ASSOCIATION AS OF JANUARY 1, 1955

(Except for the offices of President and Secretary-Treasurer, this list includes only the names of those who have held office since January 1, 1947. For information about preceding years, consult the American Mathematical Monthly for December 1951.)

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L. L. DINES	1945–1947	A. W. RECHT	1946–1948
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E. S. HAMMOND	1946–1949	C. W. WATKEYS	1946–1948
E. H. HANSON	1950–1953	MARIE J. WEISS	1950–1952
M. R. HESTENES	1950–1952	F. B. WILEY	1949–1952
E. H. C. HILDEBRANDT	1947–1950	K. P. WILLIAMS	1945–1947
D. L. HOLL	1953–1954	W. L. WILLIAMS	1946–1949

CALENDAR OF FUTURE MEETINGS

Thirty-sixth Summer Meeting, University of Michigan, Ann Arbor, Michigan, August 29–30, 1955.

Thirty-ninth Annual Meeting, Rice Institute, Houston, Texas, December 30, 1955.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Duquesne University, Pittsburgh, Pennsylvania, May 7, 1955.

ILLINOIS, Monmouth College, Monmouth, May 13–14, 1955.

INDIANA, Butler University, Indianapolis, May 7, 1955.

IOWA, St. Ambrose College, Davenport, April 15–16, 1955.

KANSAS, Fort Hays Kansas State College, Hays, March 26, 1955.

KENTUCKY, Georgetown College, Georgetown, April 30, 1955.

LOUISIANA-MISSISSIPPI

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Morgan State College, Baltimore, Maryland, April 16, 1955.

METROPOLITAN NEW YORK, Queens College, Flushing, New York, April 30, 1955.

MICHIGAN, Michigan State College, East Lansing, March 26, 1955.

MINNESOTA, College of St. Teresa, Winona, Minnesota, May 7, 1955.

MISSOURI, University of Kansas City, April 22, 1955.

NEBRASKA, University of Nebraska, Lincoln, April 23, 1955.

NORTHERN CALIFORNIA

OHIO, Ohio State University, Columbus, April 23, 1955.

OKLAHOMA

PACIFIC NORTHWEST, University of British Columbia, Vancouver, June 17, 1955.

PHILADELPHIA

ROCKY MOUNTAIN, University of Wyoming, Laramie, April 22–23, 1955.

SOUTHEASTERN, Tennessee Polytechnic Institute, Cookeville, March 11–12, 1955.

SOUTHERN CALIFORNIA, Santa Monica City College, March 12, 1955.

SOUTHWESTERN, University of New Mexico, Albuquerque, Spring, 1955.

TEXAS, Abilene Christian College, Abilene, April 22–23, 1955.

UPPER NEW YORK STATE, University of Buffalo, May 14, 1955.

WISCONSIN, Cardinal Stritch College, Milwaukee, May 7, 1955.

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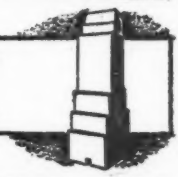

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